



SUBJECT

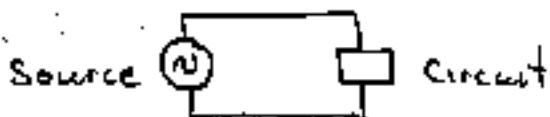
NAME

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REVISION DATE

Terminology & Conventions

Terminology & Conventions



Sinusoidal Source $V(t) = V_0 \cos(\omega t + \phi)$

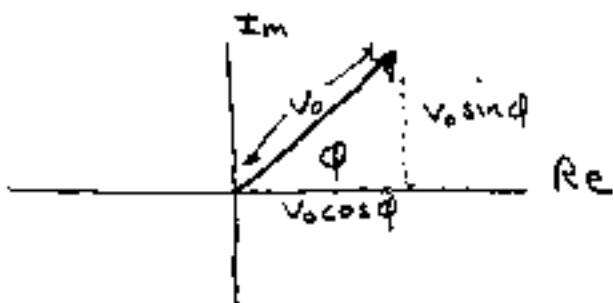
$$V(t) = \operatorname{Re} \{ V_0 e^{j(\omega t + \phi)} \}$$

where $j = \sqrt{-1}$

$$V(t) = \operatorname{Re} \{ V_0 e^{j\phi} e^{j\omega t} \}$$

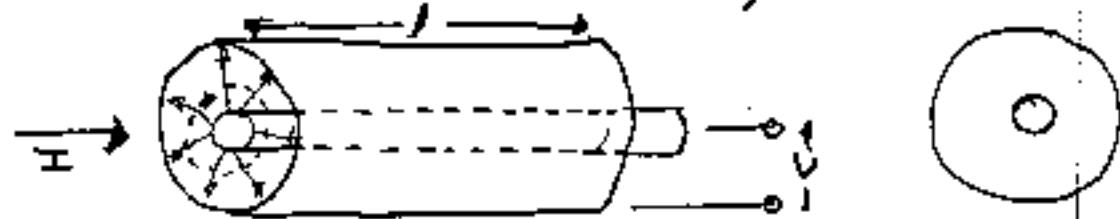
$V_0 e^{j\phi}$ is a complex phasor

$$V_0 e^{j\phi} = V_0 \cos \phi + j V_0 \sin \phi$$



In this course, we will assume all sources are sine waves. We will describe circuits with complex phasors. The time vary answer is found by multiplying by $e^{j\omega t}$ and taking the real part.

TEM Transmission Line Theory



Charge on inner conductor:

$$q = C l V$$

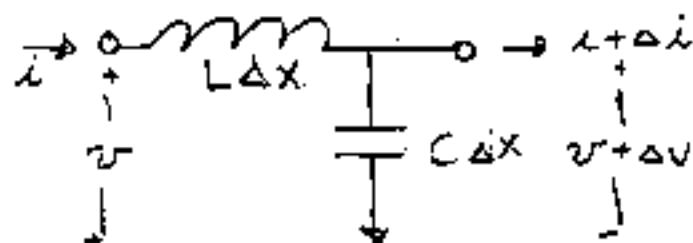
where C is the capacitance per unit length

Azimuthal magnetic flux:

$$\Phi = L l I$$

where L is the inductance per unit length.

Electrical model of a small section of coax.



Voltage drop along the inductor

$$v - (v + \Delta v) = L \Delta x \frac{di}{dt}$$

Current flowing thru the capacitor

$$i + \Delta i = i - C \Delta x \frac{dv}{dt}$$

Limit as $\Delta \rightarrow 0$

$$-\frac{\partial v}{\partial x} = L \frac{di}{dt}$$

$$-\frac{di}{dx} = C \frac{dv}{dt}$$

Solutions are traveling waves

$$v(t, x) = v^+(t - \frac{x}{vel}) + v^-(t + \frac{x}{vel})$$

- + indicates wave traveling in the + direction
- indicates wave traveling in the - direction

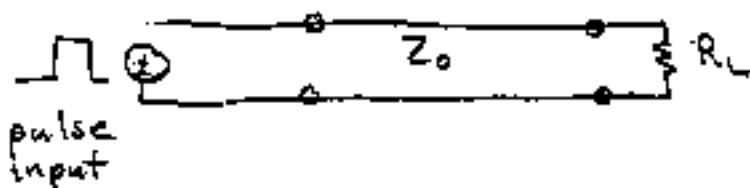
$$i = \frac{1}{Z_0} v^+(t - \frac{x}{vel}) - \frac{1}{Z_0} v^-(t + \frac{x}{vel})$$

where vel is the phase velocity of the wave

$$vel = \frac{1}{\sqrt{\mu\epsilon}}$$

$$= \frac{1}{\sqrt{\mu\epsilon}} \text{ for a TEM wave}$$

$Z_0 = \sqrt{\frac{\mu}{\epsilon}}$ has units of Ω 's
and is called the characteristic impedance



Pulse travels down the transmission line as a forward going current & voltage wave only. However, when pulse reaches load resistor we must have:

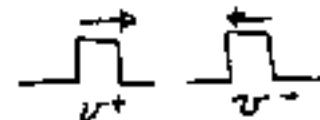
$$\frac{v^+}{Z_0} = R_L = \frac{v^+ + \frac{2E^-}{Z_0}}{\frac{2E^+}{Z_0} - \frac{2E^-}{Z_0}}$$

So a reverse wave v^- & i^- must be created to satisfy the boundary condition imposed by the load resistor.

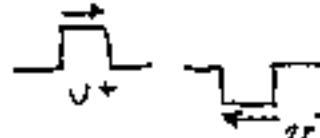
$$\frac{v^-}{v^+} = \frac{R_L - Z_0}{R_L + Z_0} = \Gamma \quad (\text{reflection coeff})$$

Three cases:

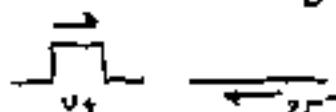
$$R_L = \infty \text{ (open)} \quad \Gamma = +1$$



$$R_L = 0 \text{ (short)} \quad \Gamma = -1$$



$$R_L = Z_0 \quad \Gamma = 0$$



\therefore A transmission line terminated with its characteristic impedance looks the same as an infinitely long line.

Sinusoidal Waves

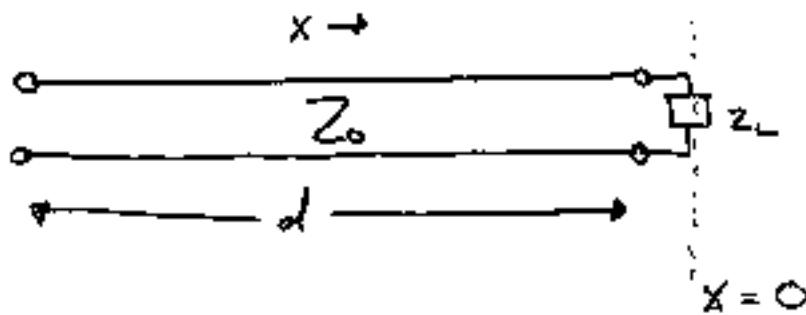
$$v^+ = V^+ \cos(\omega t - Bx) = \operatorname{Re}\{V^+ e^{-jBx} e^{j\omega t}\}$$

where $\frac{\omega}{B} = \text{vel} = \text{phase velocity}$

$$B = \frac{2\pi f}{\text{vel}} \quad \text{or} \quad B = \frac{2\pi}{\lambda}$$

By using sine waves we can now use complex impedances such as

$$Z_{\text{capacitor}} = \frac{1}{j\omega C} \quad Z_{\text{inductor}} = j\omega L$$



$$\text{at } x=0 \text{ we have } V^- = \Gamma V^+ = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\begin{aligned} V &= V^+ e^{-j\beta x} + \Gamma V^+ e^{+j\beta x} \\ &= V^+ (1 + \Gamma - \Gamma) e^{-j\beta x} + \Gamma V^+ e^{+j\beta x} \\ &\approx V^+ (1 - \Gamma) e^{-j\beta x} + 2V^+ \Gamma \left(\frac{e^{-j\beta x} + e^{+j\beta x}}{2} \right) \\ &= V^+ (1 - \Gamma) e^{-j\beta x} + 2V^+ \Gamma \cos \beta x \end{aligned}$$

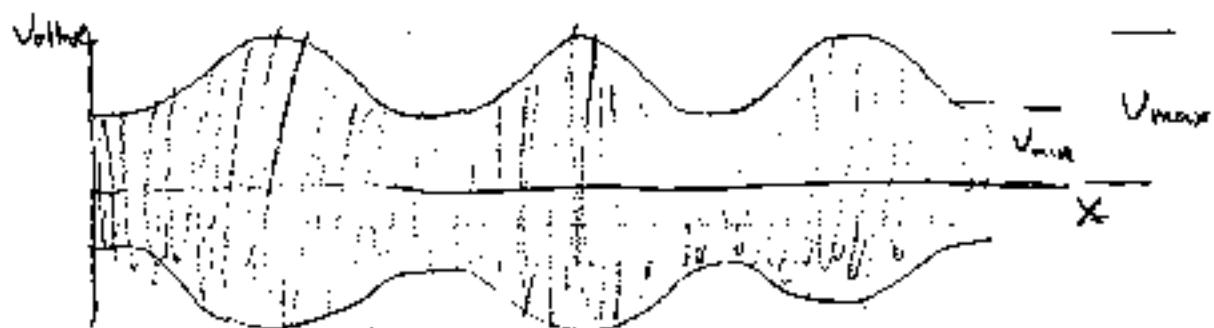
↑ travelling wave ↑ standing wave

$$\frac{V_{max}}{V_{min}} = \frac{V^+ (1 + |\Gamma|)}{V^+ (1 - |\Gamma|)} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

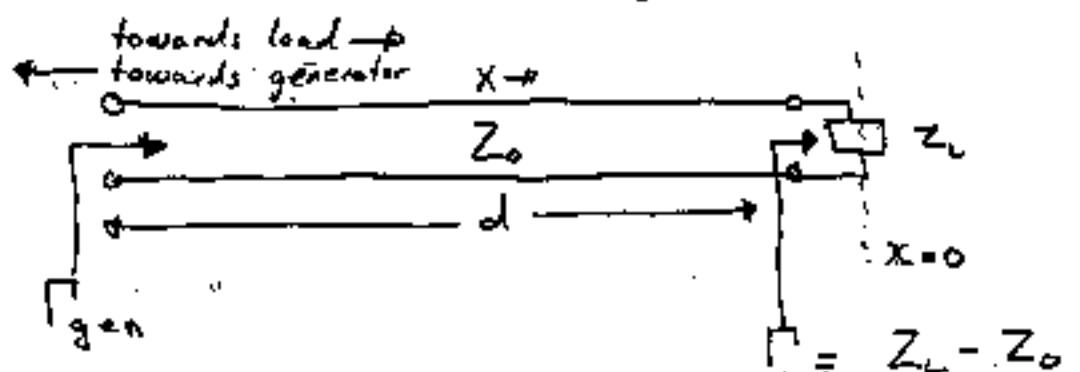
$$= \text{VSWR}$$

↑
 ↓
 ↓
 ↓
 ↓

$$\text{VSWR} \geq 1$$



Reflection Coefficient along a transmission line



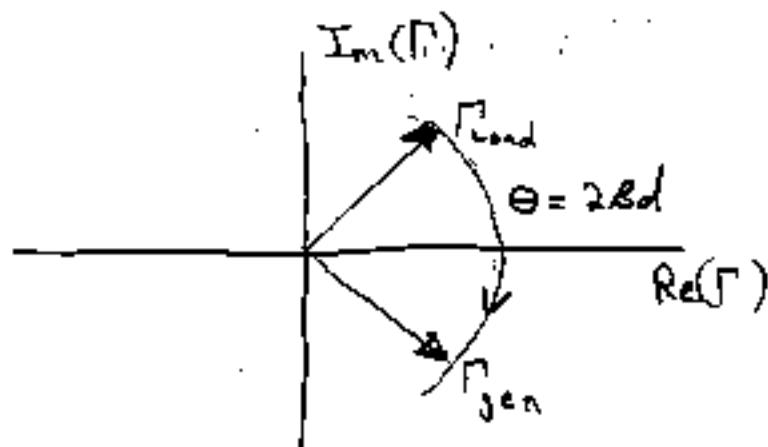
$$\Gamma_{\text{load}} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$V = V^+ e^{-j\beta x} + \Gamma_{\text{load}} V^+ e^{+j\beta x}$$

at $x = -d$ (at generator what is ratio of reverse wave to forward wave)

$$\left| \frac{V_{\text{forward}}}{V_{\text{reverse}}} \right|_{\text{gen}} = \left| \frac{\Gamma_{\text{load}} V^+ e^{-j\beta d}}{V^+ e^{+j\beta d}} \right| = \left| \Gamma_{\text{load}} e^{-j2\beta d} \right| \quad (z = -d)$$

$$\Gamma_{\text{gen}} = \Gamma_{\text{load}} e^{-j2\beta d}$$



There is still a one to one correspondence of Γ with load impedance

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

$$Z = Z_0 \left(\frac{1 + \Gamma}{1 - \Gamma} \right)$$

$$Z = Z_0 \left(\frac{1 + \Gamma_0 e^{-j\beta d}}{1 + \Gamma_0 e^{j\beta d}} \right)$$

For $\Gamma = 1$ (open) $Z = -jZ_0 \cot \beta d$

$$\begin{aligned} \text{for } \beta d \ll 1 \quad Z &\approx -j \frac{Z_0}{\beta d} \\ &= -j \frac{\sqrt{\frac{L}{C}}}{d \omega \sqrt{LC}} \\ &= \frac{j}{\omega C d} \quad (\text{capacitive}) \end{aligned}$$

$$\text{for } \beta d = \frac{\pi}{2} \quad \text{or } d = \frac{\lambda}{4} = \dots$$

$$Z = 0$$

∴ an open looks like a short $1/4$ wavelength away from the open.

For $\Gamma = -1$ (short) $Z = jZ_0 \tan \beta d$

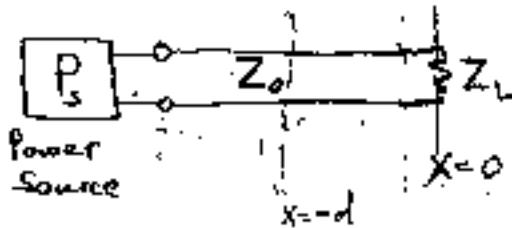
$$\begin{aligned} \text{for } \beta d \ll 1 \quad Z &\approx jZ_0 \beta d \\ &= j \sqrt{\frac{L}{C}} d \omega \sqrt{LC} \\ &= j \omega L d \quad (\text{inductive}) \end{aligned}$$

$$\text{for } \beta d = \frac{\pi}{2} \quad \text{or} \quad d = \frac{\lambda}{4}$$

$$Z = \infty$$

i.e. a short looks like an open $\lambda/4$ wavelength away from the short.

Incident and reflected power.



Voltage along the transmission line at $x = -d$

$$V(x = -d) = V_{inc} e^{j\beta d} + \Gamma V_{inc} e^{-j\beta d}$$

Current at $x = -d$

$$I(x = -d) = \frac{V_{inc}}{Z_0} e^{j\beta d} - \Gamma \frac{V_{inc}}{Z_0} e^{-j\beta d}$$

The rate of Energy flowing thru the plane

at $x = -d$ is equal to

$$P = \frac{1}{2} \operatorname{Re} \left\{ V I^* \right\}$$

$$= \frac{1}{2} \frac{V_{inc}^2}{Z_0} \operatorname{Re} \left\{ 1 - |\Gamma|^2 + \Gamma e^{-j\beta d} - \Gamma^* e^{j\beta d} \right\}$$

form only an
imaginary number

$$P = \frac{V_{inc}^2}{2Z_0} (1 - |\Gamma|^2)$$

$$P = \frac{V_{inc}^2}{2 Z_0} - \frac{|F|^2 V_{inc}^2}{2 Z_0}$$

↗ ↗
incident power reflected power

* Remember: Power does not flow, Energy does.
 The forward & reverse traveling waves are power orthogonal so that the net rate of energy transfer is equal to the difference in power of the individual waves.

To maximize the power absorbed by the load we want

$$F = 0$$

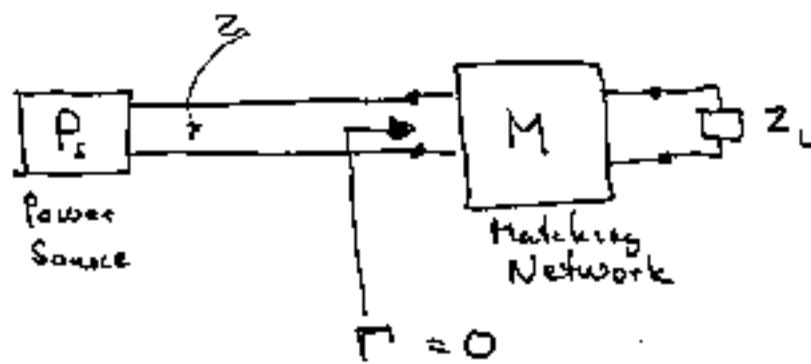
or

$$Z_L = Z_0$$

When $Z_L = Z_0$ we say that the load is matched to the transmission line.

Smith Charts

What if our load cannot be made equal to Z_0 for some other reasons? We need to build a matching network so that the source effectively sees a match load.



Normalized impedance

$$\gamma = \frac{Z}{Z_0} = r + jx$$

The reflection coefficient is a complex number

$$\Gamma = u + jv$$

$$\gamma = \frac{1 + \Gamma}{1 - \Gamma}$$

$$r = \frac{1 - u^2 - v^2}{(1-u)^2 + v^2}$$

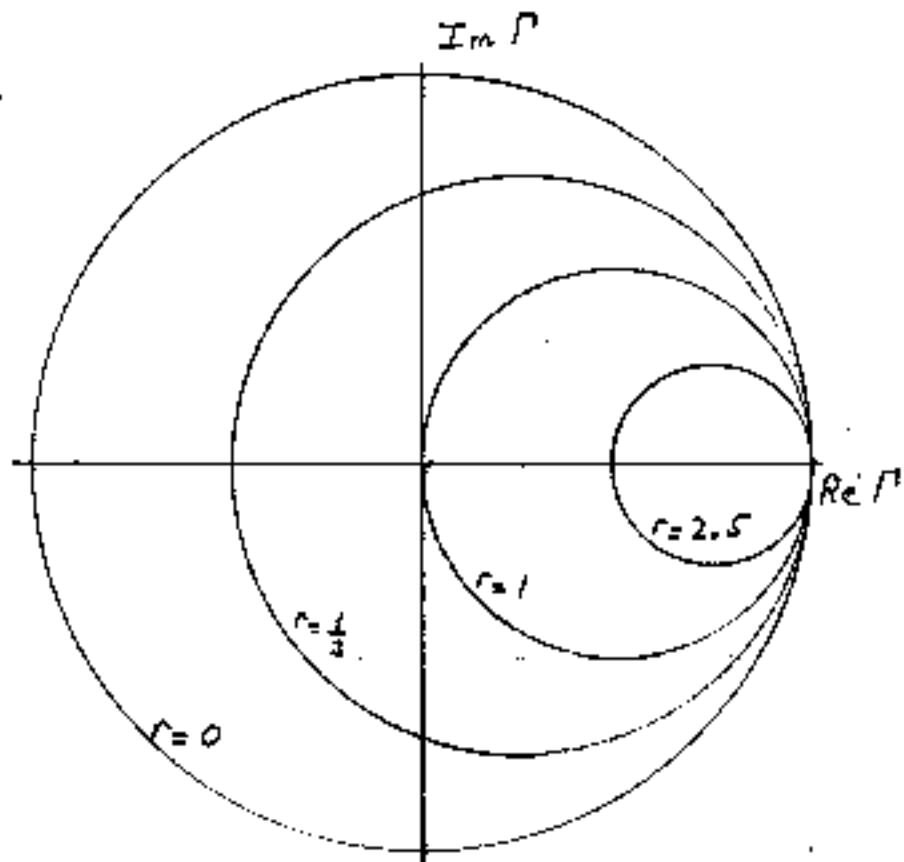
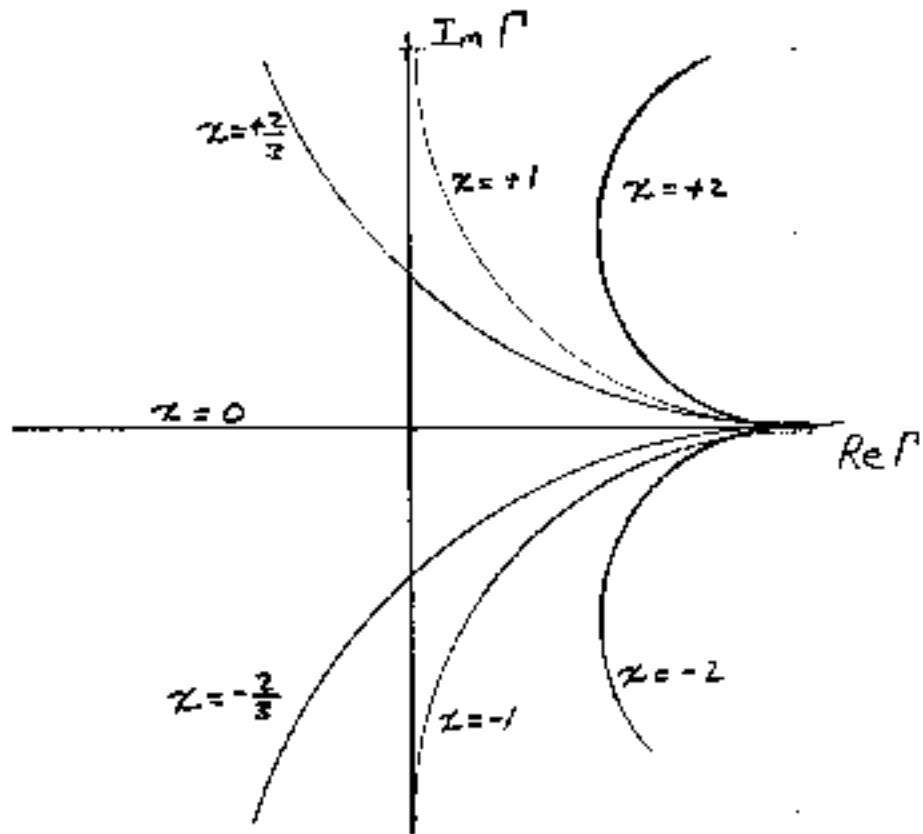
$$x = \frac{2v}{(1-u)^2 + v^2}$$

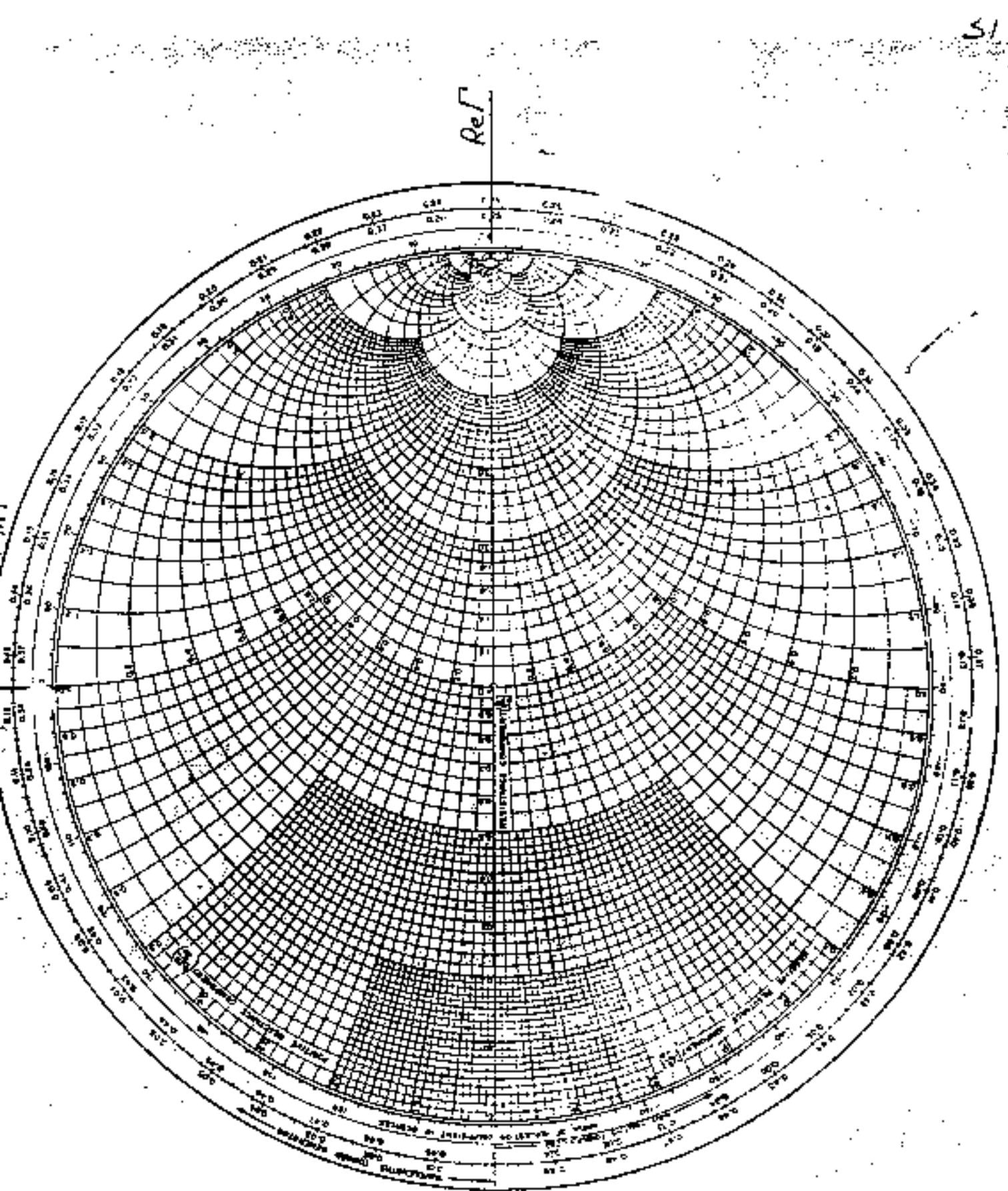
OR:

$$\left(u - \frac{\Gamma}{1+\Gamma}\right)^2 + v^2 = \frac{1}{(1+\Gamma)^2}$$

$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$

These are eqns. for circles on the u, v plane.

Γ circles χ circles



① Given $\Gamma_L = .5 \angle 45^\circ$ $Z_0 = 50\Omega$

what is Z_L

$$\begin{aligned} Z_L &= 50\Omega (1.35 + j1.35) \\ &= 67.5\Omega + j67.5\Omega \end{aligned}$$

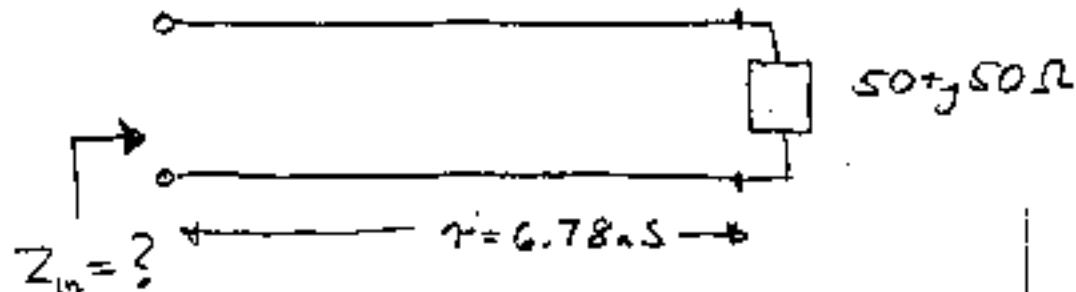
② Given $Z_L = 15\Omega - j25\Omega$ $Z_0 = 50\Omega$

what is Γ_L

$$Z_L = .3 - j.5$$

$$\Gamma_L = .618 \angle -124^\circ$$

③ $Z_0 = 50\Omega$



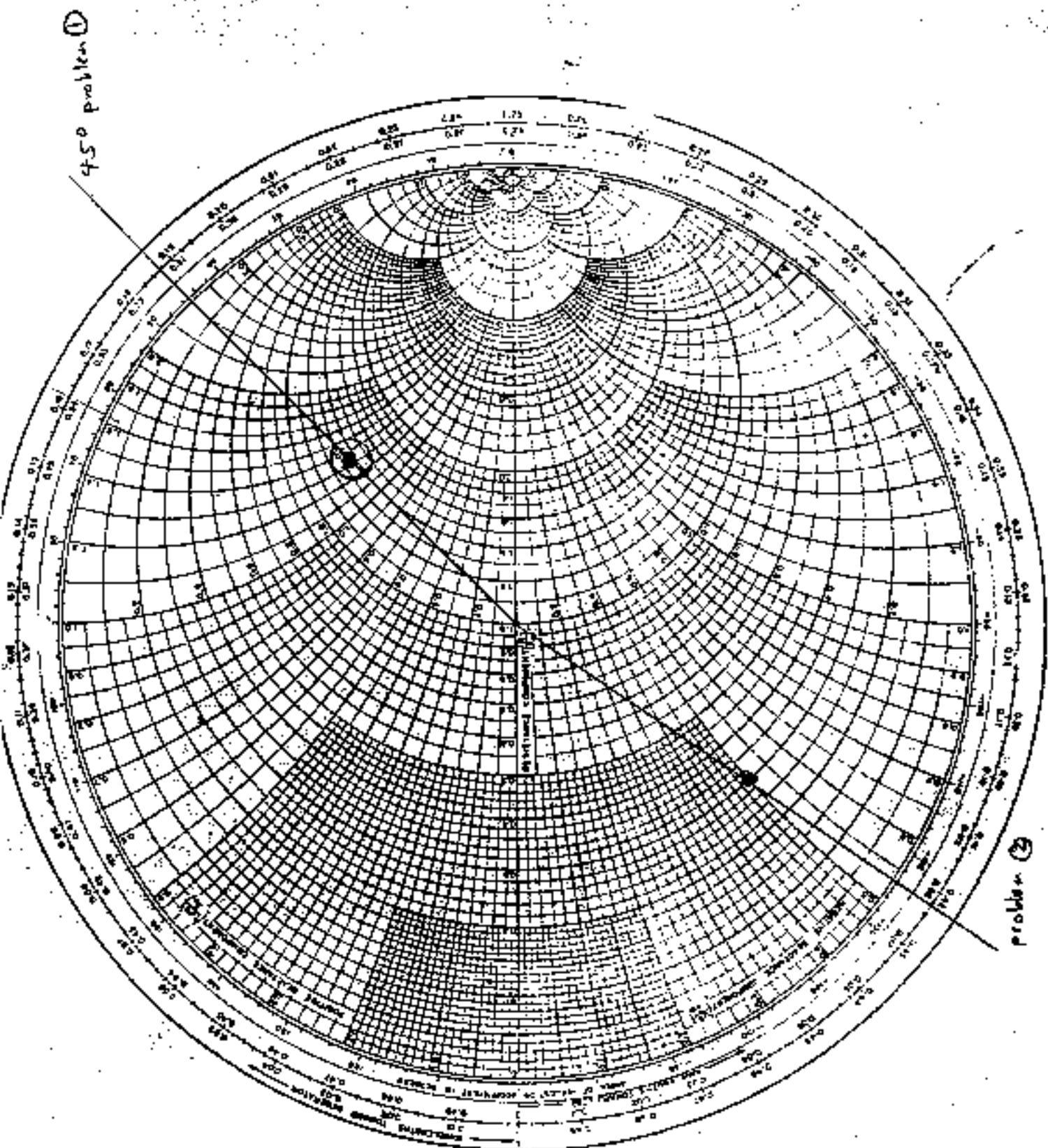
at 50MHz

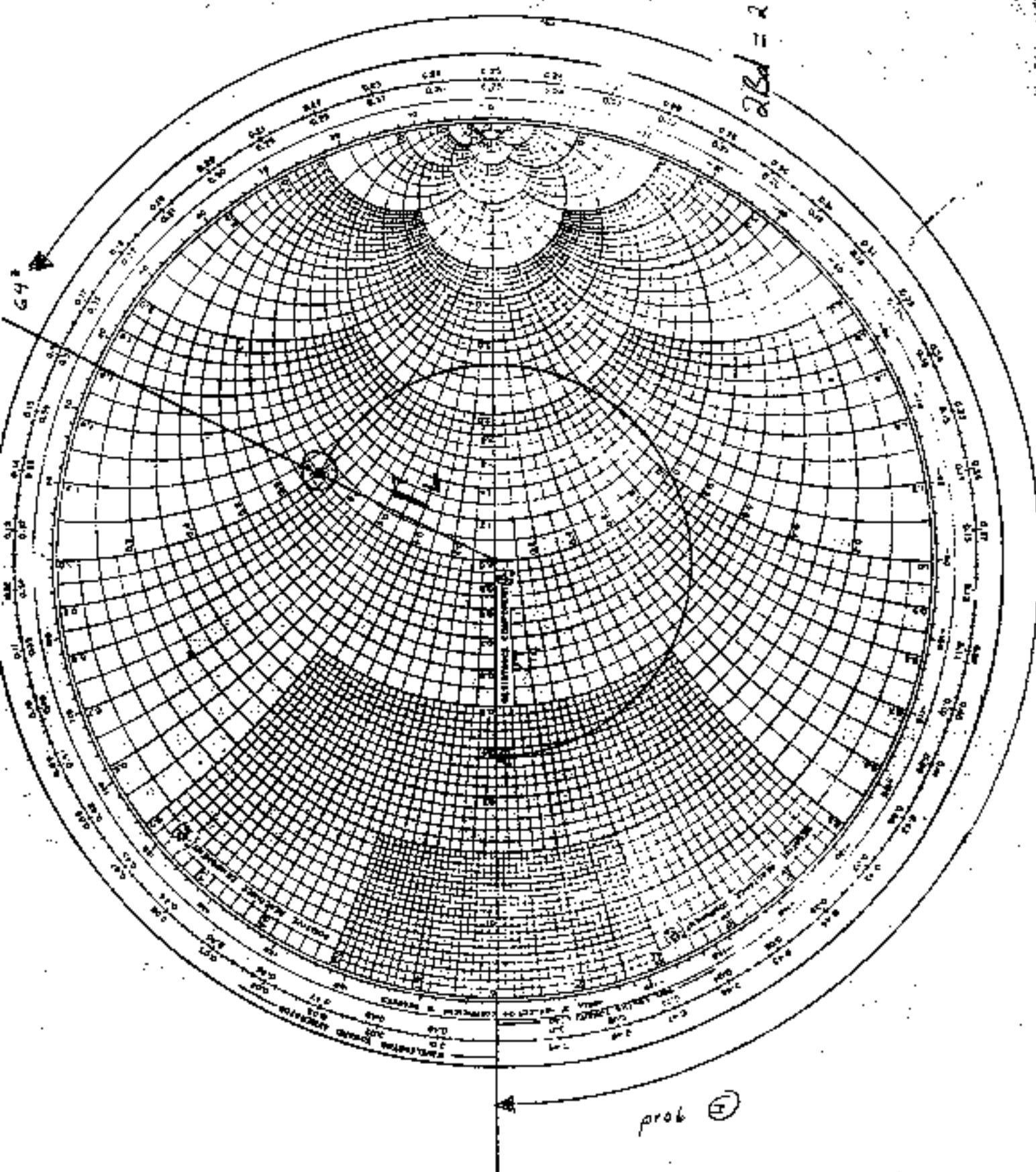
$$Z_L = 1 + j1$$

$$\Gamma_L = .445 \angle 64^\circ$$

$$\Gamma_{in} = \Gamma_L e^{-j2Bd}$$

$$2Bd = \frac{2\pi f}{v_{el}} \cdot d$$





$$2Bd = 2 \cdot 360^\circ \cdot f \cdot \tau \quad \left(\frac{d}{\text{vel}} = \tau \right)$$

$$= 244^\circ$$

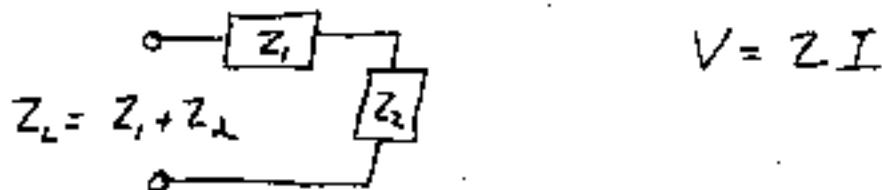
$$P_{in} = .445 \angle -180^\circ$$

$$Z_{in} = 50\Omega (0.38 + j0)$$

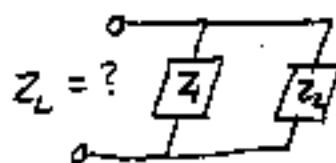
$$Z_{in} = 19 \Omega$$

Admittance

Impedance is nice to work with series configurations. For example:



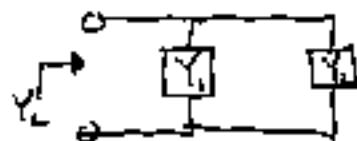
But not so nice for parallel configurations



$$Z_L = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

For parallel loads it is easier to work with admittance

$$I = YV$$



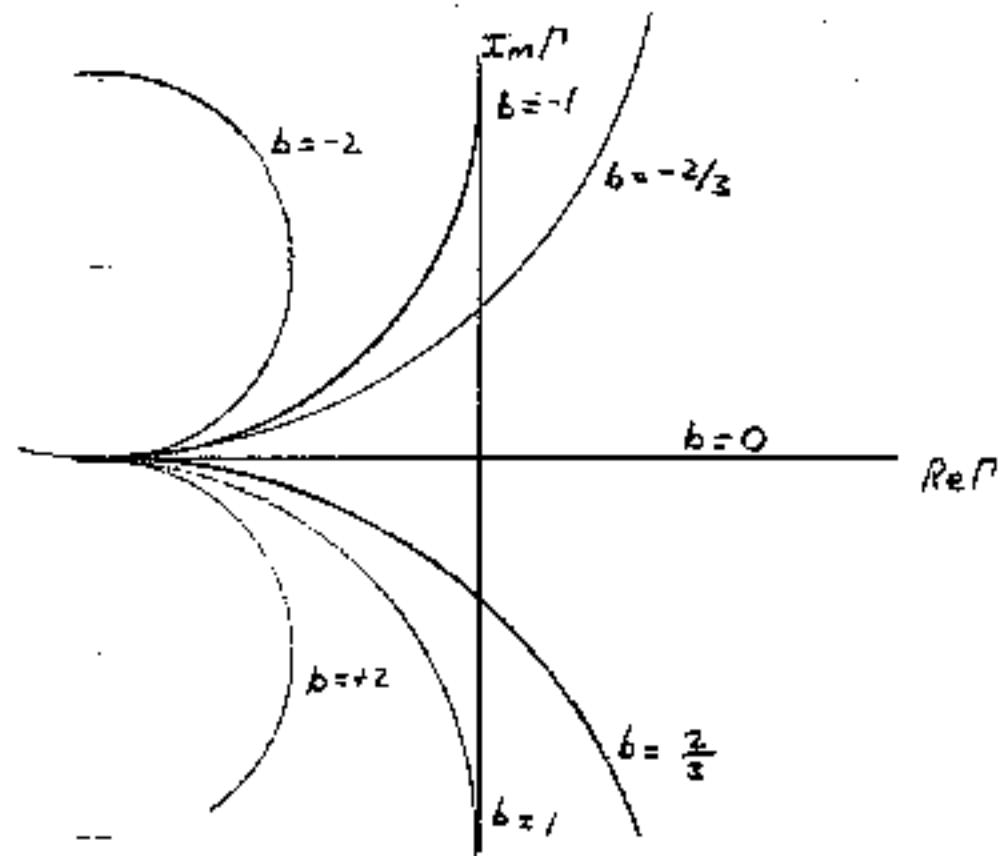
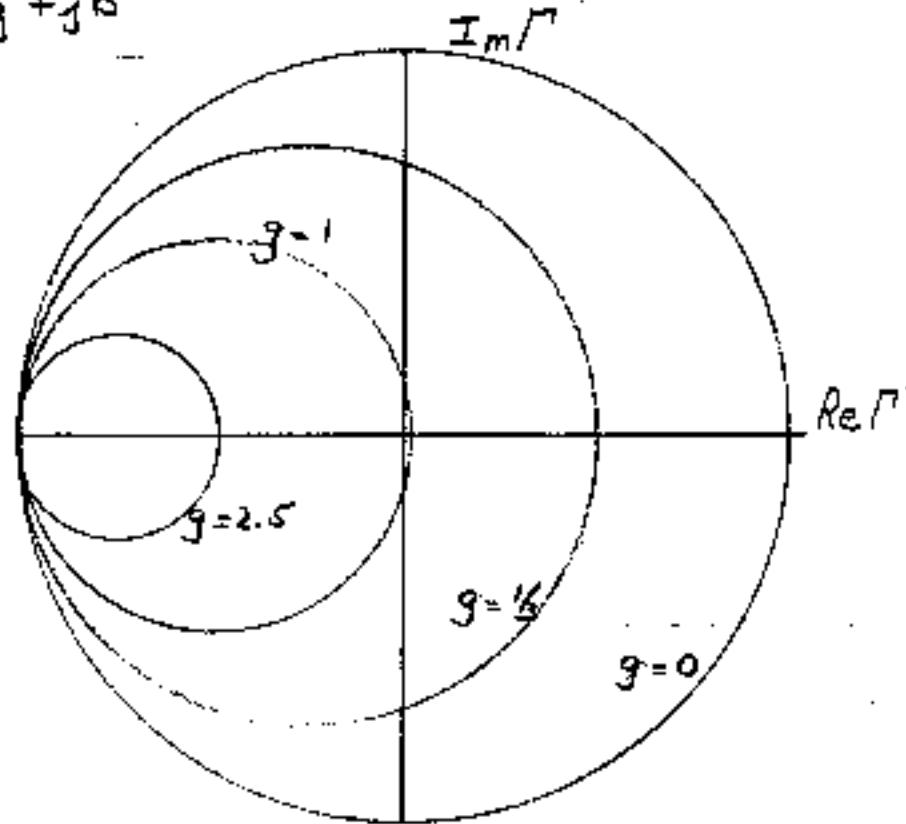
$$Y_1 = \frac{1}{Z_1}$$

$$Y_2 = \frac{1}{Z_2}$$

$$Y_L = Y_1 + Y_2$$

$$\gamma = \frac{\gamma}{\gamma_0} = g + jb$$

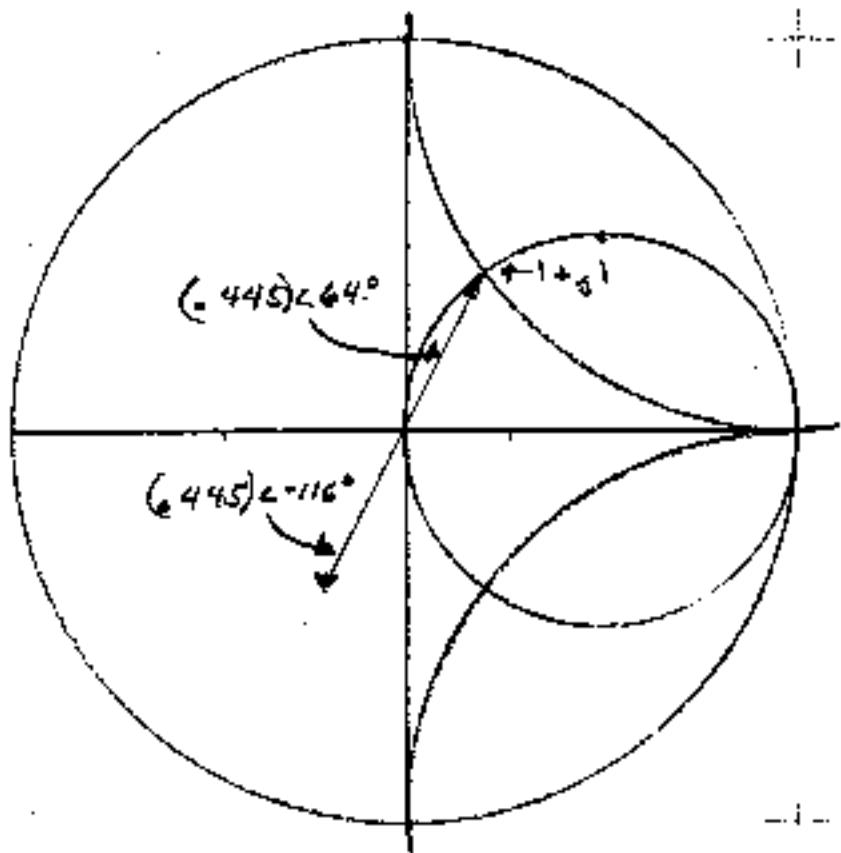
$$\gamma_0 = \frac{1}{Z_0}$$



We see that the admittance Smith Chart is the impedance Smith Chart rotated 180° . We could use TWO charts one for impedance & one for admittance

OR:

We could use one chart & just flip the reflection coefficient vector 180°



Example 1

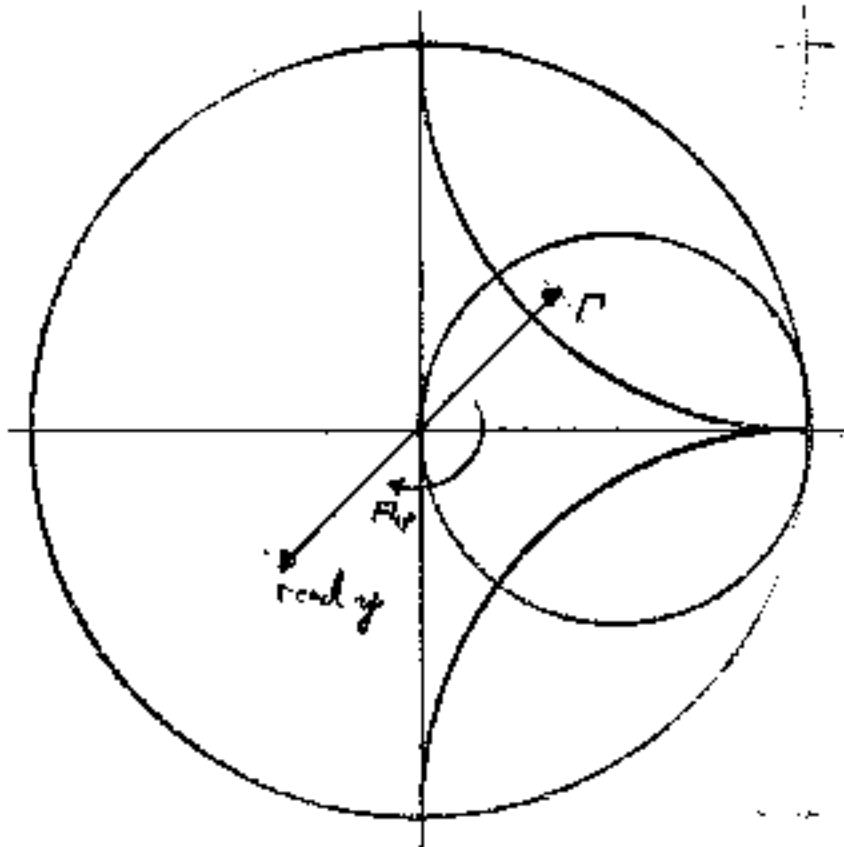
If $y = 1 + j1$ what is Γ ?

a) plot $1 + j1$ on chart

$$\text{Vector} = .445 \angle 64^\circ$$

b) flip vector by 180°

$$\Gamma = .445 \angle -116^\circ$$

Example 2

If $\Gamma = 0.5 \angle 45^\circ$ $Z_0 = 50\Omega$

What is Y

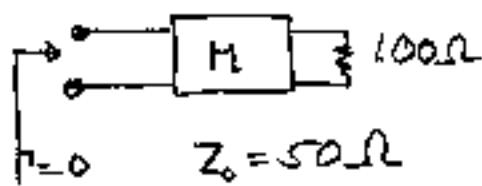
a) Plot Γ

b) Flip Γ by 180°

c) read coordinate = $.38 - j.36$

d) $Y = \frac{1}{50\Omega} (.38 - j.36) = (7.6 - j7.2) \times 10^{-3} \text{ mhos}$

Matching Example:

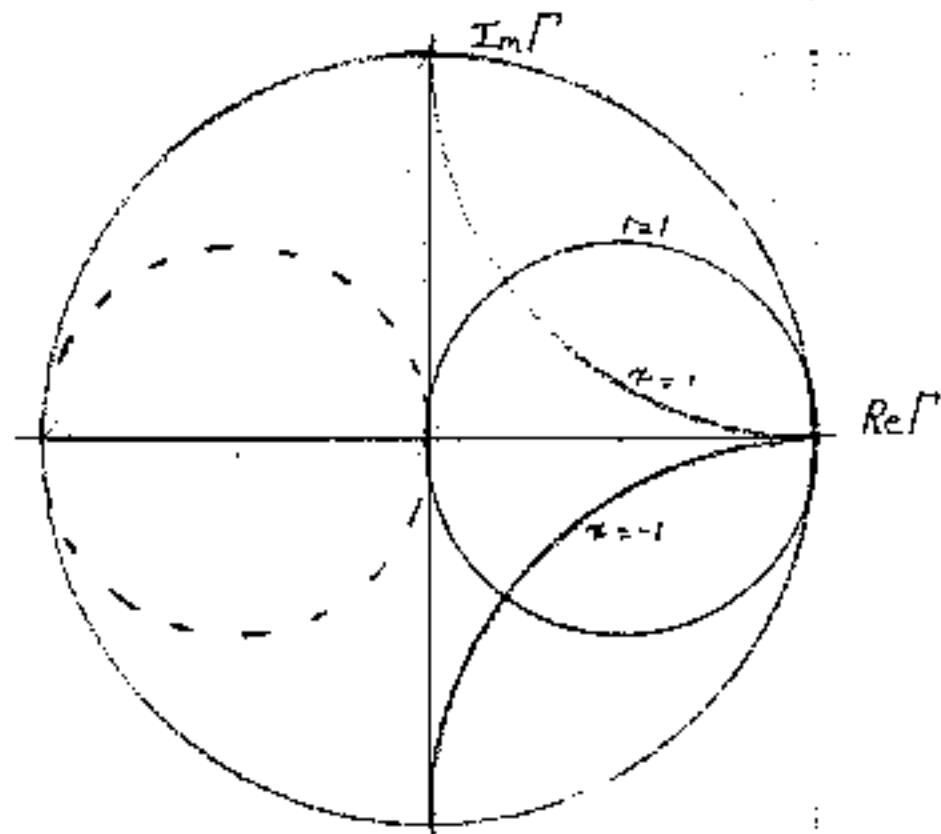


Match a 100Ω load

to a 50Ω transmission
line system @ 100 MHz

We want to use only lossless elements such as capacitors & inductors so we don't dissipate any power in the matching network.

To make things easier, let's draw in an extra circle on the Smith Chart, the mirror image of the $r=1$ circle



1) We need to go from $y = 2+j0$ to $z = 1+j0$ on Smith chart. We won't get any closer by adding a series impedance to the load, so we need to flip over to admittance which is at $0.5+j0$.

2) Now add positive imaginary admittance to the load until we reach the the mirror image circle

$$j_b = j \cdot 5$$

$$j(0.5) \frac{1}{j_0C} = 2\pi 100 \text{MHz} C$$



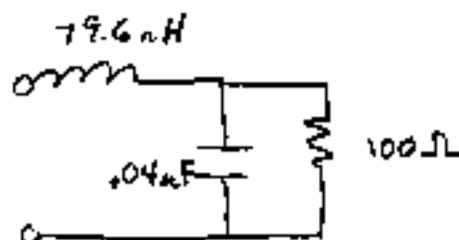
$$C = 16 \text{ pF}$$

3) Now flip back to impedance
We now land on the $r=1$ circle at $x=-1$

4) Add positive imaginary admittance to get us to $z = 1+j0$

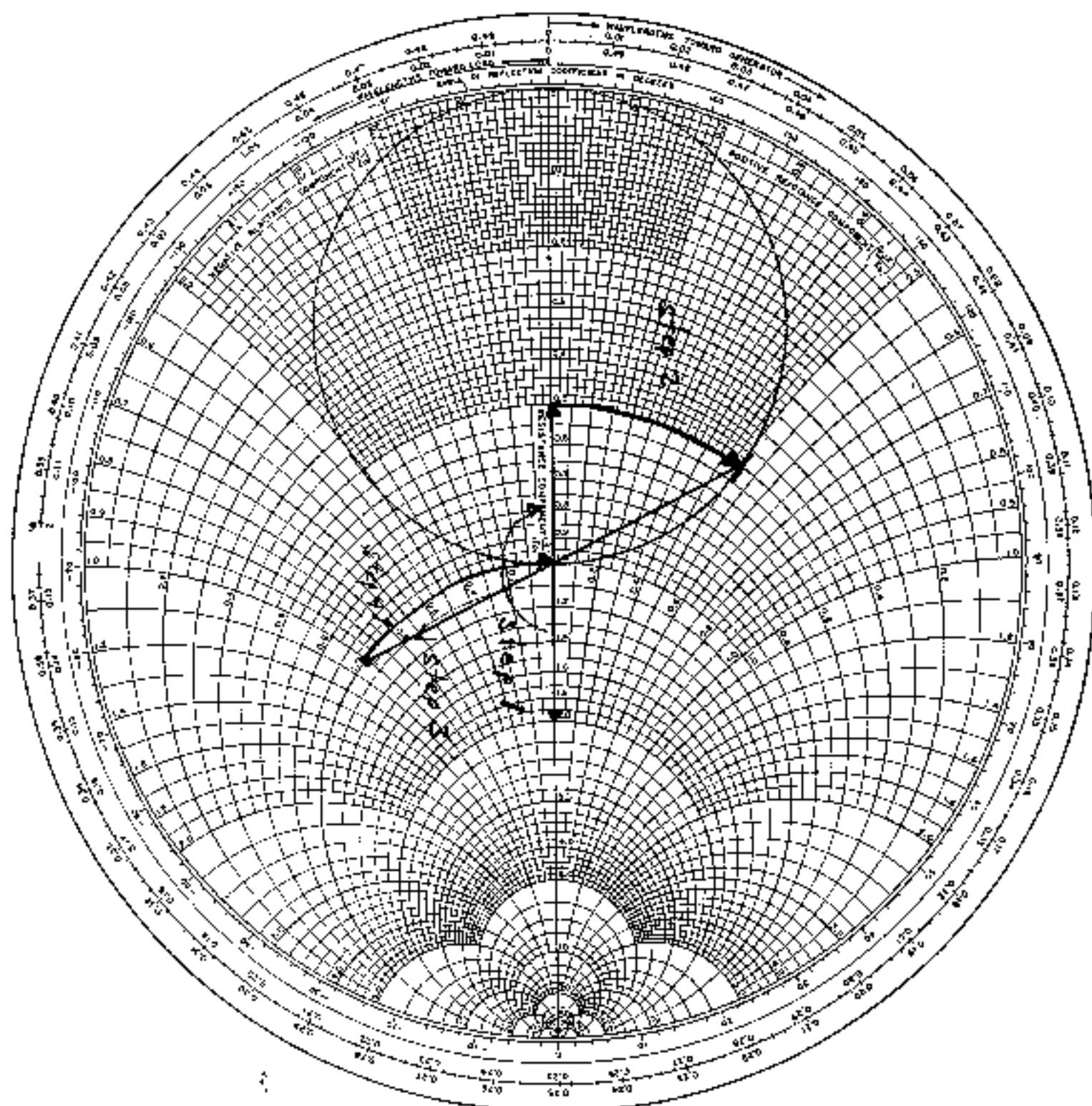
$$j_x = -j1$$

$$(j1)50\Omega = 2\pi 100 \text{MHz} L$$

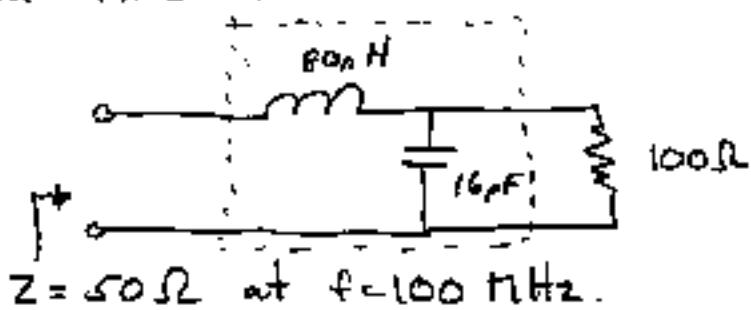


$$L = 79.6 \text{ nH}$$

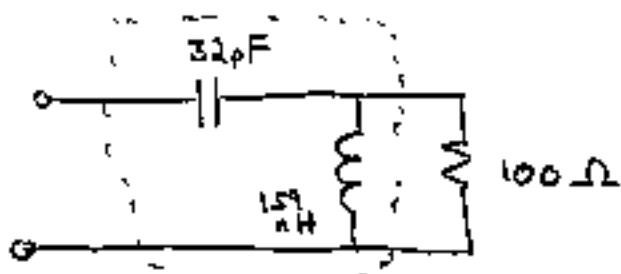
THE EMECO INC.
HILLSDIDE, N.J.



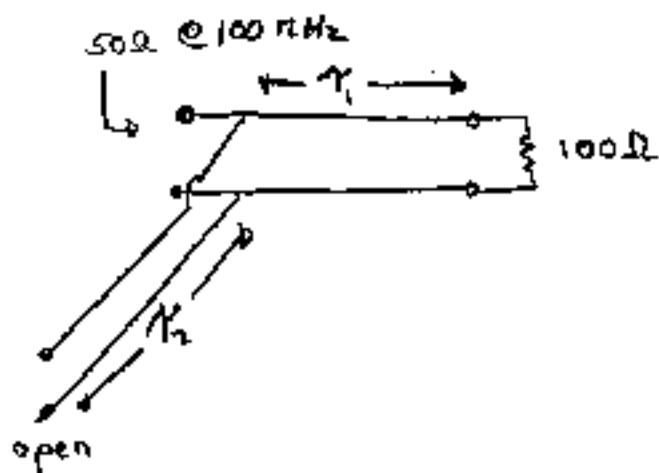
Final Answer



This answer would have also worked:



Single stub tuner



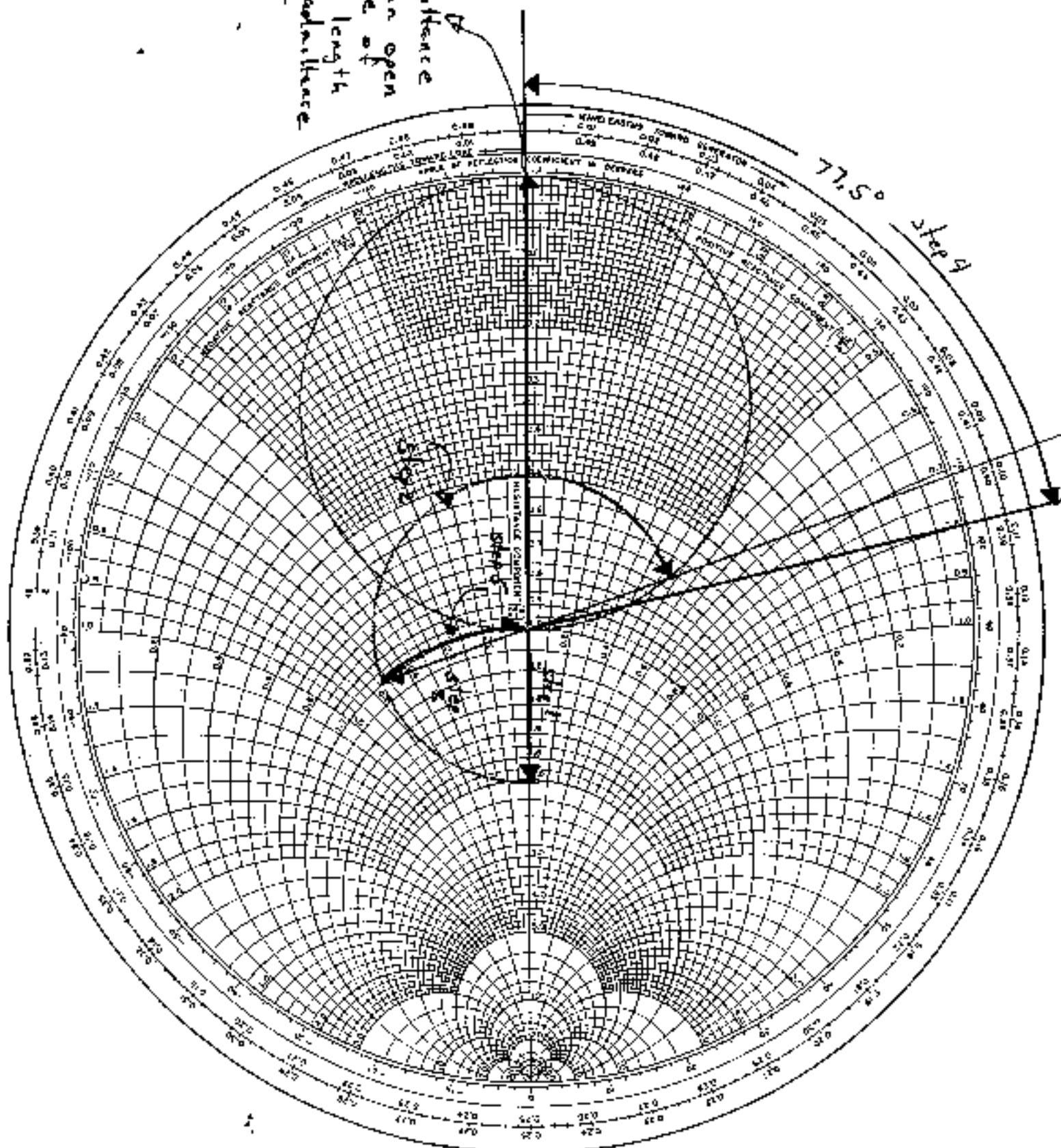
- plot reflection coefficient
- Rotate reflection coefficient by traveling an angle of $\Theta = 251^\circ$ towards the generator

$$\Theta = 2 \cdot 360^\circ - 100 \text{ MHz} \cdot \tau_1$$

$\tau_1 = 3.5 \text{ ns}$ to land on mirror image circle

THE ENCLIO CO., INC.
HILLSIDE, N. J.

Admittance
of an open
line of
zero length
on admittance
chart



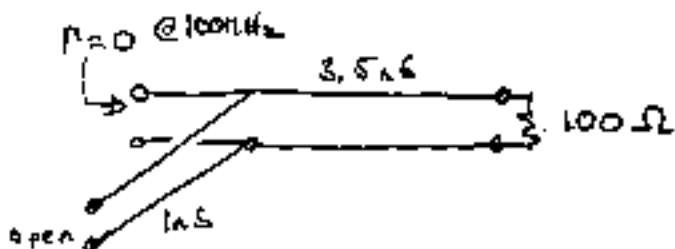
- 3) Flip over by 180° for admittance
 At this point we have an ^{normalized} admittance of
 $1 - j \cdot 8$ admittance

We need to add an normalized admittance of $+ j \cdot 8$ to transmission line

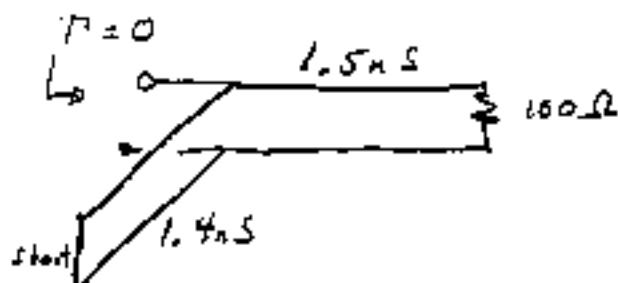
4. An open circuit line of zero length has an admittance of $j0$. If we rotate by 72.5° the line will have an admittance of
- $$72.5^\circ = 2 \cdot 360^\circ / 100\text{MHz} \gamma_2$$

$$\gamma_2 = 1 \text{ nS}$$

5.



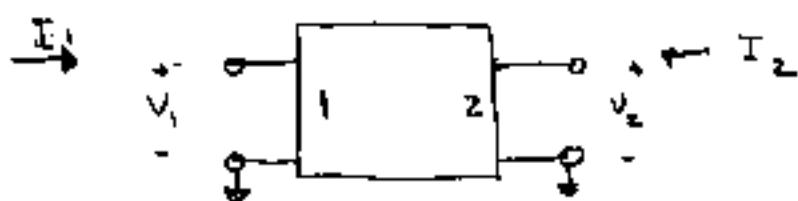
This answer would have worked as well



Scattering Parameters (S parameters)

We have only worried about reflection so far. What about transmission?

Consider a device which has two ports (or spigots)



The device can be characterized by 2x2 matrix.

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

$$[V] = [Z] [I]$$

$$\text{but since } V_i = V_i^+ + V_i^-$$

$$Z_i I_i = V_i^+ - V_i^-$$

where V_i^\pm stand for forward and reverse waves travelling into part i.

We can characterize the Circuit by the voltage waves

$$V_i^+ = S_{11} V_1^+ + S_{12} V_2^+$$

$$V_2^+ = S_{21} V_1^+ + S_{22} V_2^+$$

$$[v^-] = [S][v^+]$$

The S matrix is called the scattering matrix. This is actually the un-normalized scattering matrix.

For convenience define

$$a_i = \frac{v_i^+}{\sqrt{2}Z_{o,i}}$$

$$b_i = \frac{v_i^-}{\sqrt{2}Z_{o,i}}$$

where $Z_{o,i}$ is the characteristic impedance of the transmission line connecting port i .

$|a_i|^2$ = forward power into port i

$|b_i|^2$ = reflected power out of port i .

$$b_1 = s_{11}a_1 + s_{12}a_2$$

$$b_2 = s_{21}a_1 + s_{22}a_2$$

$$[b] = [s][a]$$

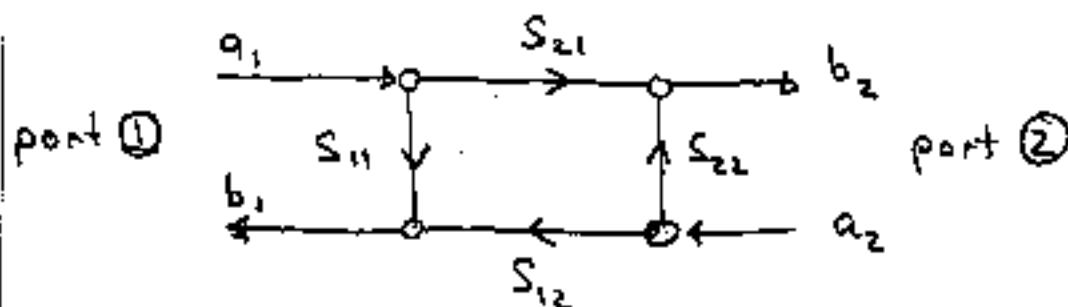
where $[s]$ is the power normalized scattering matrix.

where

$$S_{2j} = \sqrt{\frac{Z_{0j}}{Z_{0i}}} S_{2j}$$

Normalized s parameters are most commonly used. i.e. $Z_{01} = Z_{02}$

The s parameters can be drawn pictorially



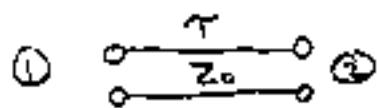
S_{11} and S_{22} can be thought of as reflection coefficients

S_{21} and S_{12} can be thought of as transmission coefficients

Remember, the s parameters are complex numbers where the angle corresponds to a phase shift between the input & output waves.

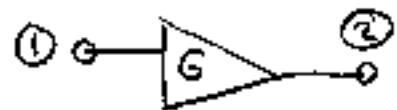
Some common S matrices

Transmission line



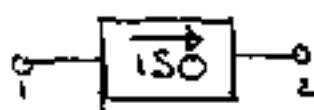
$$[S] = \begin{bmatrix} 0 & e^{-j\omega f r} \\ e^{-j360^\circ f r} & 0 \end{bmatrix}$$

As amplifier.



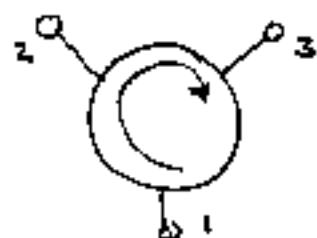
$$S = \begin{bmatrix} 0 & 0 \\ G & 0 \end{bmatrix}$$

An isolator



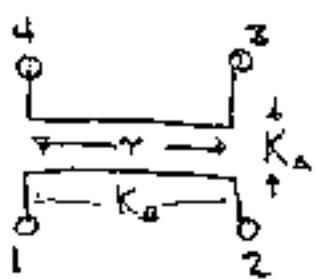
$$S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

A circulator



$$S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

A directional coupler



$$S = \begin{bmatrix} 0 & K_A e^{-j2\pi f t} & jK_A e^{j2\pi f t} & 0 \\ K_A e^{-j2\pi f t} & 0 & 0 & jK_A e^{j2\pi f t} \\ jK_A e^{j2\pi f t} & 0 & 0 & K_A e^{-j2\pi f t} \\ 0 & jK_A e^{-j2\pi f t} & K_A e^{j2\pi f t} & 0 \end{bmatrix}$$

Two Useful Laws

A. Lorentz reciprocity

If the device is made out of linear materials i.e. resistors, inductors, capacitors metal etc.

Then $[S]$ is symmetric

$$[S]_{\text{Transpose}} = [S]$$

$$\text{or } S_{ij} = S_{ji} \quad \text{for } j \neq i$$

This is the same as saying that the transmitting pattern of an antenna is the same as the receiving pattern of the antenna.

From the above examples

Reciprocal Devices { Transmission line
Directional Coupler

Non-reciprocal Devices { Amplifier
Isolator
Circulator

B. Lossless Devices

The S parameter matrix of a lossless device is Unitary

$$[S^*]_{\text{Transpose}} [S] = [I]$$

To check Loss, Sum up the magnitudes of any row or column, the sum must be 1.

$$1 = \sum_{j=1}^n |S_{ij}|^2 \text{ for all } i$$

or

$$1 = \sum_{i=1}^n |S_{ij}|^2 \text{ for all } j$$

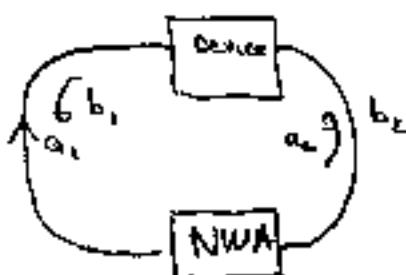
for the device to be lossless.

Lossless devices
from above examples:

{ Transmission line
Isolator
Circulator
Direction coupler if
 $|K_{pl}|^2 + |K_{sl}|^2 = 1$

Network Analyzers

Network analyzers measure the s parameters as a function of frequency



$$S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2=0}$$

$$S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2=0}$$

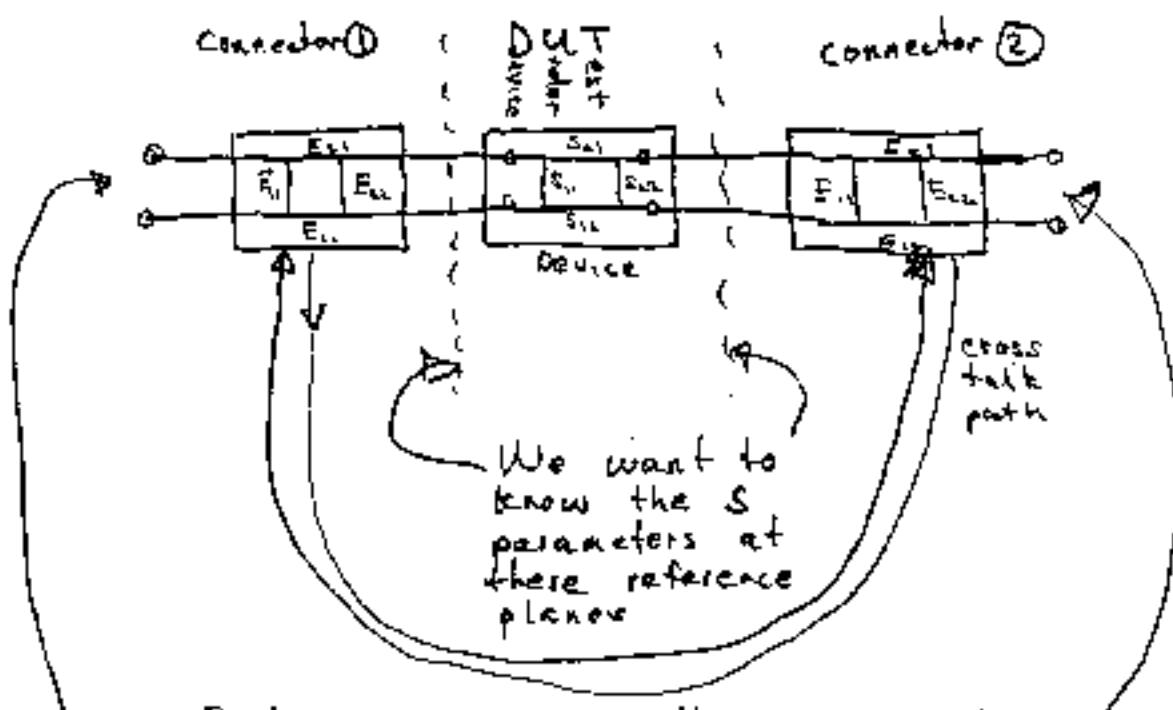
To make $a_2 = 0$ the NWA has to be matched at port 2 because $b_2 \neq 0$ for $S_{21} \neq 0$

$$S_{22} = \left. \frac{b_2}{a_2} \right|_{a_1=0}$$

$$S_{12} = \left. \frac{b_1}{a_2} \right|_{a_1=0}$$

Calibration

To measure the pure S parameters of a device we need to eliminate the effects the cables, connectors, etc. attaching the device to the network analyzer.

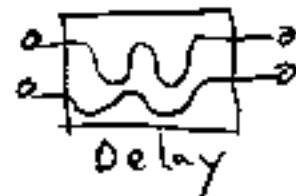
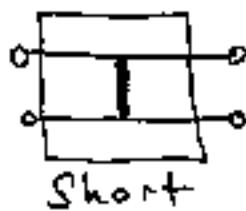


But we measure the S parameters at these reference planes

There are 10 unknowns in the connectors
We need 10 independent measurements to eliminate these unknowns.

Came up with calibration standards.
We place the standards in where the DUT would be and measure the S-parameters of the standards. Since we know a priori the S parameters of the standards, we can

determine the unknown S parameters of the connectors. Once these connector parameters are known, they can be mathematically eliminated from the measurements once the DUT is placed back in the measuring fixture. Since we get to measure 4 S parameters for each calibration standard, we need at least 3 independent standards. One possible set is



This is known as calibration or de-embedding.

Group Velocity and phase velocity

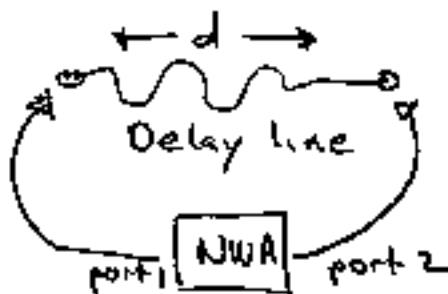
A pure sine wave can be written as

$$V = V_0 e^{j(\omega t - \beta z)}$$

The phase velocity is equal to

$$v_{\text{phase}} = \frac{c}{\beta}$$

We can measure the phase velocity by setting up an input wave to delay line and measuring the phase difference between the input and output wave.



We measure the angle of S_{21} at a given freq.

$$\arg(S_{21}) = \beta d = \Theta \quad \text{--- (1)}$$

$$v_{\text{phase}} = \frac{\text{freq} \cdot d}{\Theta/360^\circ}$$

This is the phase velocity of a single wave. Suppose we wish to measure the velocity of a pulse thru a device.

The pulse is composed of many frequencies. Each of the frequencies might be travelling at a different phase velocity. The group velocity is

$$v_{\text{group}} = \frac{d\omega}{d\beta}$$

The group delay is the time it takes the pulse to travel a distance d

$$\tau_{\text{group}} = \frac{d}{v_{\text{group}}}$$

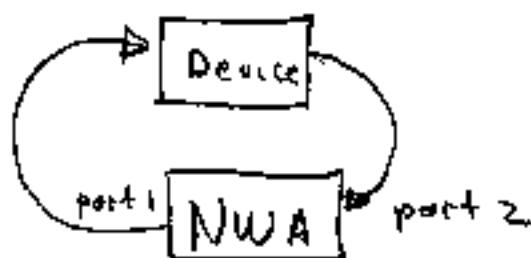
$$\tau_{\text{group}} = d \frac{\partial \beta}{\partial \omega}$$

but βd is the phase of the wave

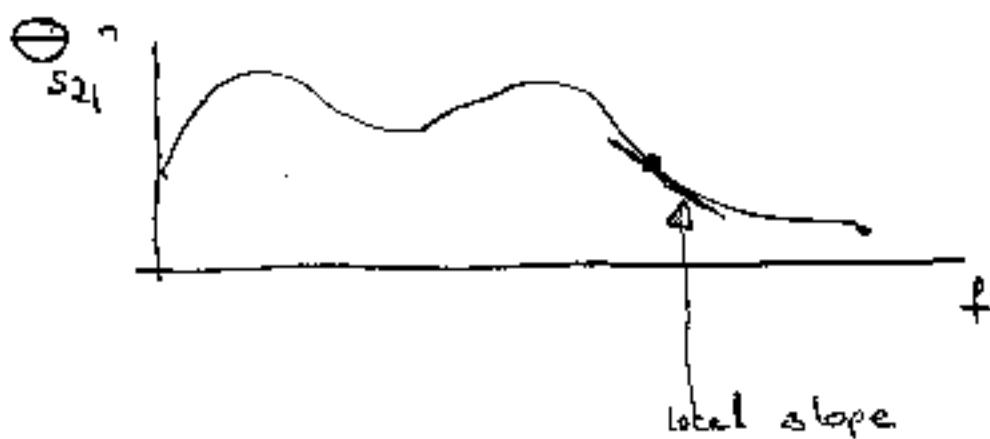
$$\tau_{\text{group}} = \frac{\partial \theta}{\partial \omega}$$

$$\tau_{\text{group}} = \frac{1}{360^\circ} \frac{\partial \theta}{\partial f}$$

To measure the group delay of a device, Measure S21



The slope of the angle of S_{21} is equal to the group delay



$$\tau_{\text{group}}(f) = \frac{1}{360^\circ} \frac{\partial(\Theta_{S_{21}}(f))}{\partial f}$$



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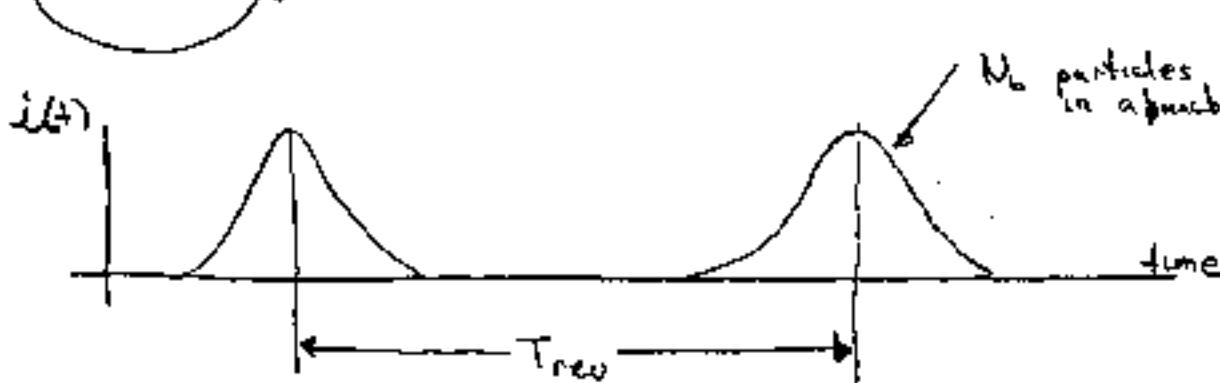
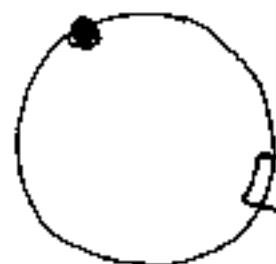
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Beam Signals for Circular Machines

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Single bunch in a machine.

Bunch shape = $f(t)$ where $\int f(t) dt = 1$

$$i(t) = \sum_{n=-\infty}^{\infty} q N_b f(t - n T_{rev})$$

This is a periodic series

$$i_b(t) = \sum_{m=-\infty}^{\infty} c_m e^{j m \omega_{rev} t}$$

$$\text{where } \omega_{rev} = \frac{2\pi}{T_{rev}}$$

$$c_m = \frac{q N_b}{T_{rev}} \int_{T_{rev}/2}^{T_{rev}/2} f(t) e^{-j m \omega_{rev} t} dt$$



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Beam Signals for Circular Machines

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Note $C_0 = \frac{q N_b}{T_{\text{rev}}} \quad (\text{DC current})$

Since $f(t)$ is a real function

$$C_{-m} = C_m^*$$

Fourier Spectrum of Beam current.

$$i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) e^{i\omega t} d\omega$$

$$I(\omega) = \int_{-\infty}^{\infty} i(t) e^{-i\omega t} dt$$

Since $\delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega - \omega')t} dt$

$$I_b(\omega) = 2\pi \sum_{m=-\infty}^{\infty} c_m \delta(\omega - m\omega_m)$$

But spectrum analyzers do not measure currents & voltages. They measure power!

(Actually FFT boxes measure voltages & currents)

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Power Spectral Density.

Time averaged power

$$\langle p(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} R \int_{-T/2}^{T/2} x(t) \cdot x(t) dt$$

Since

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) e^{i\omega t} d\omega$$

and $x(t)$ is real

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I^*(\omega) e^{-i\omega t} d\omega$$

$$\begin{aligned} \langle p(t) \rangle &= \lim_{T \rightarrow \infty} \frac{1}{T} R \int_{-T/2}^{T/2} dt \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_1 I(\omega_1) e^{i\omega_1 t} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_2 I(\omega_2)^* e^{-i\omega_2 t} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} R \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_1 I(\omega_1) \int_{-\infty}^{\infty} d\omega_2 I^*(\omega_2) \frac{1}{2\pi} \int_{-T/2}^{T/2} dt e^{i(\omega_1 - \omega_2)t} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} R \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_1 I(\omega_1) \int_{-\infty}^{\infty} d\omega_2 I^*(\omega_2) \delta(\omega_1 - \omega_2) \\ &= \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{R}{2\pi T} |I(\omega)|^2 d\omega \end{aligned}$$

Power Spectral Density

$$S(\omega) = \lim_{T \rightarrow \infty} \frac{R}{2\pi T} |I(\omega)|^2$$

$$\langle p(t) \rangle = \int_{-\infty}^{\infty} S(\omega) d\omega$$



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Power Spectral Density.

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Spectrum analyzers measure $S(\omega)$!!

$$I_n(\omega) = 2\pi \sum_{m=-\infty}^{\infty} c_m \delta(\omega - m\omega_{rev})$$

$$S_n(\omega) = \lim_{T \rightarrow \infty} \frac{R}{2\pi T} (2\pi)^2 \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} c_m c_{m'}^* \delta(\omega - m\omega_{rev}) \delta(\omega - m'\omega_{rev})$$

Since δ functions don't overlap, for $m \neq m'$

$$S_n(\omega) = \lim_{T \rightarrow \infty} \frac{R}{T} 2\pi \sum_{m=-\infty}^{\infty} |c_m|^2 (\delta(\omega - m\omega_{rev}))^2$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} (\delta(\omega - m\omega_{rev}))^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \delta(\omega - m\omega_{rev}) \left| \int_{-\pi}^{\pi} e^{j(\omega - m\omega_{rev})t} dt \right|^2$$

$$= \frac{1}{2\pi} \delta(\omega - m\omega_{rev})$$

$$S_n(\omega) = R \sum_{m=-\infty}^{\infty} |c_m|^2 \delta(\omega - m\omega_{rev})$$

Since Spectrum analyzers can't distinguish between positive and negative frequencies

$$S_n(\omega) = R \left(\frac{a}{f_{rev}} \right)^2 \delta(\omega) + 2R \sum_{m=1}^{\infty} |c_m|^2 \delta(\omega - m\omega_{rev})$$

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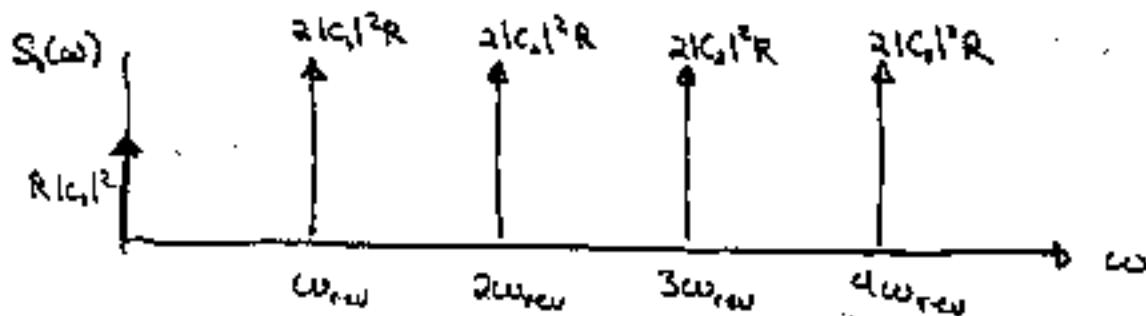
BUREAU

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Beam Spectrum

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The power contained in each revolution line

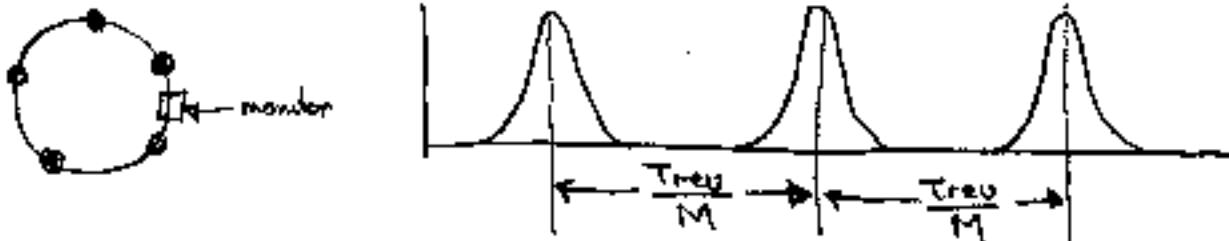
$$P_0 = R|c_0|^2 \quad m=0$$

$$P_m = 2R|c_m|^2 \quad m>0$$

Note that if $f(t) = \delta(t)$

$$c_m = \frac{qN_b}{T_{rev}} \quad \text{for all } m.$$

M equally spaced bunches in a ring



This looks exactly like 1 bunch in a machine
N times smaller

$$i_b(t) = \sum_{m=0}^{\infty} c_m e^{j m M \omega_{rev} t}$$

$$c_m = \frac{qN_b}{T_{rev}/M} \left\{ \begin{array}{l} f(t) e^{-j m M \omega_{rev} t} \\ -\frac{T_{rev}}{2\pi M} \end{array} \right.$$

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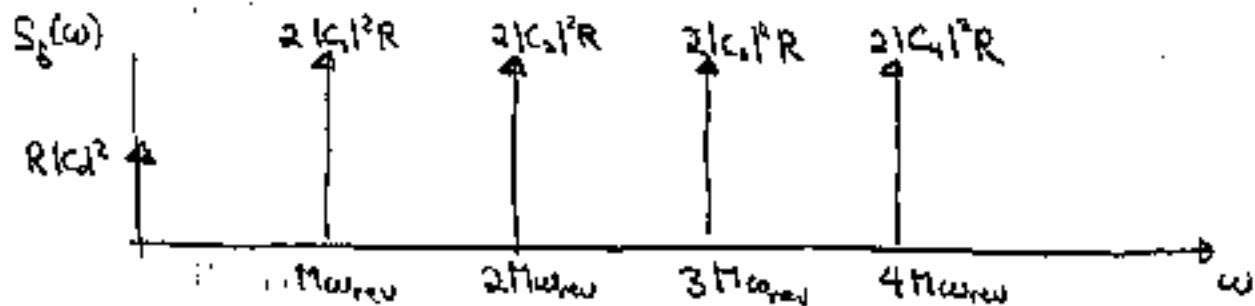
NAME

Beam Spectrum

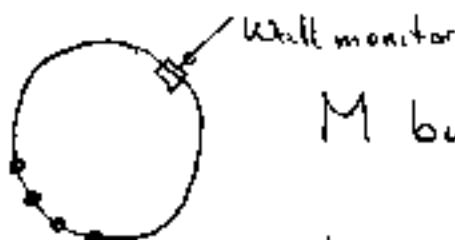
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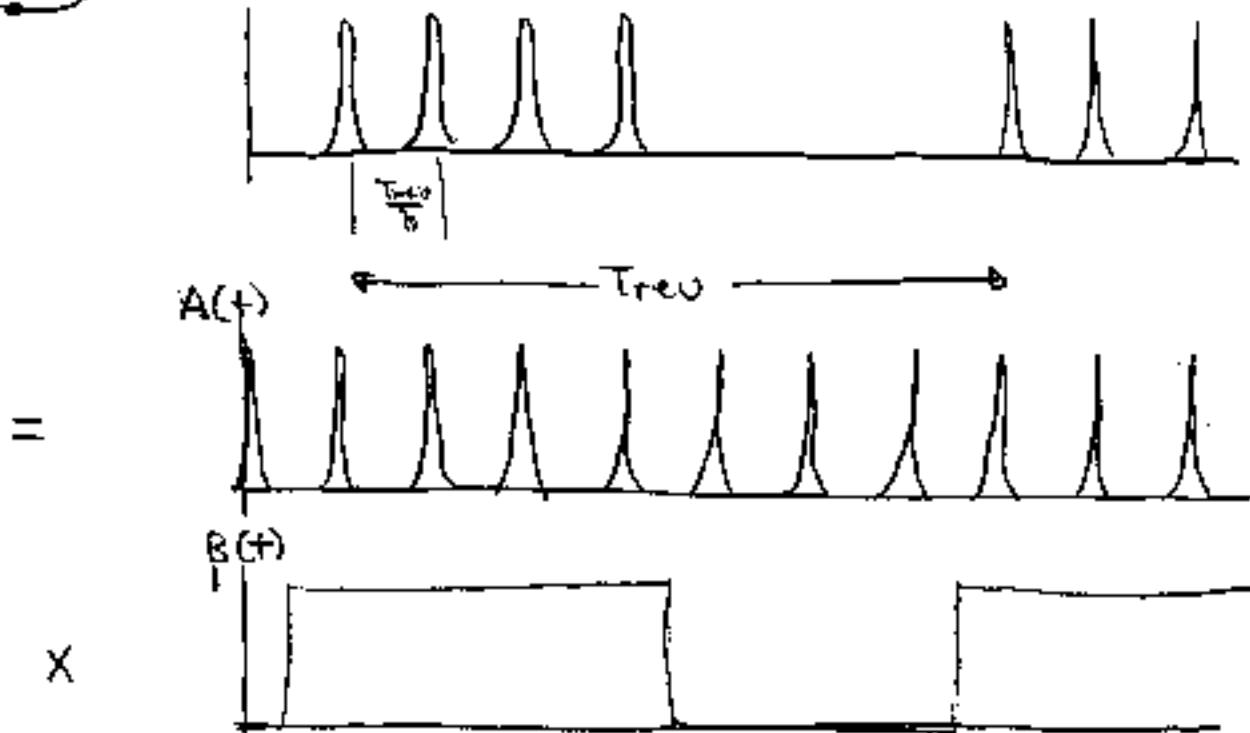
Note $C_0 = \frac{qMN}{T_{rev}}$ which is still the DC current in the machine



A burst of bunches in a machine



M bunches separated by $\frac{T_{rev}}{h}$





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Beam Spectrum

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Top wave form

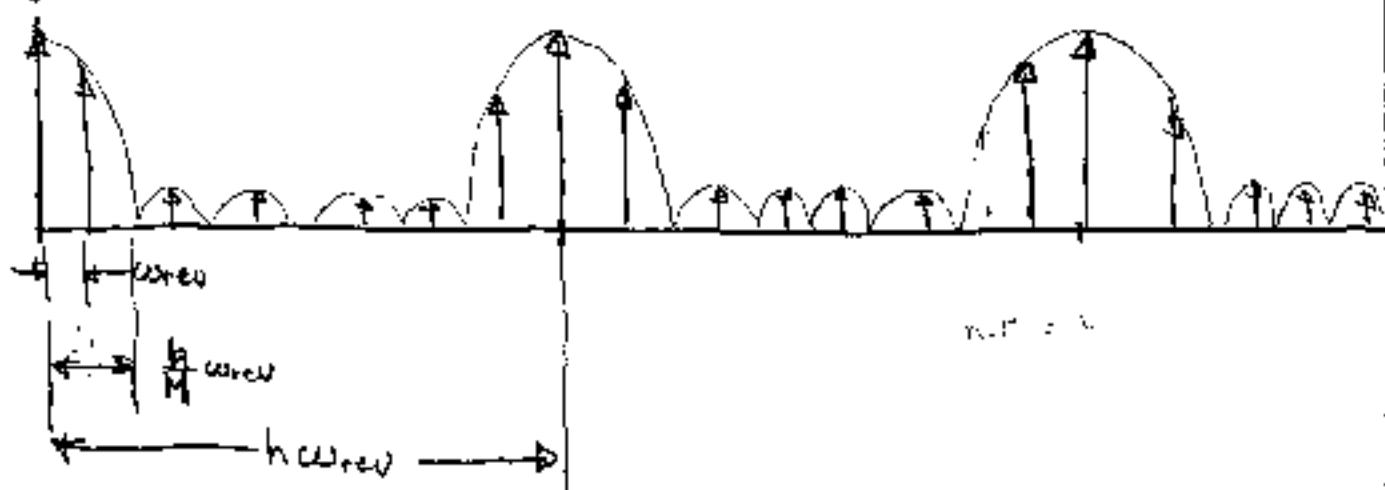
$$A(t) = q \frac{h N_b}{T_{rev}} \sum_{m=-\infty}^{\infty} d_m e^{j m h \omega_{rev} t}$$

where

$$d_m = \int_{-\frac{T_{rev}}{2}}^{\frac{T_{rev}}{2}} f(t) e^{-j m h \omega_{rev} t} dt$$

$$B(t) = \frac{M}{h} \sum_{n=-\infty}^{\infty} \text{Sa}\left(\frac{n M \pi}{h}\right) e^{j n \omega_{rev} \left(t - \frac{M T_{rev}}{h} \right)}$$

$$i_o(t) = q \frac{M N_b}{T_{rev}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_m \text{Sa}\left(\frac{n M \pi}{h}\right) e^{j \omega_{rev}(mh+n)t} e^{-j n \omega_{av} \frac{M T_{rev}}{h}}$$

 $S_o(\omega)$ 

$$\text{Sa}(x) = \frac{\sin(x)}{x}$$



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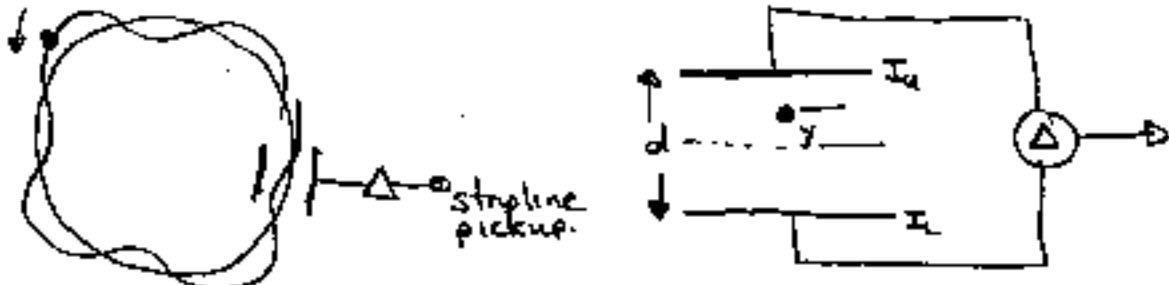
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Betatron Motion

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One particle in a machine



Current collected on upper plate

$$I_u = \frac{I_b}{2} \left(1 + \frac{2y}{d} \right)$$

$$I_L = \frac{I_b}{2} \left(1 - \frac{2y}{d} \right)$$

$$V_{out} = \frac{1}{\sqrt{2}} Z_0 (I_u - I_L) = \sqrt{2} I_b Z_0 \left(\frac{y}{d} \right)$$

$$I_b = q \sum_{n=-\infty}^{\infty} \delta(t - n T_{rev})$$

$$= \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} e^{jn\omega_{rev}t}$$

For Betatron oscillations

$$y = y_{co} + y_e \cos(Q \omega_{rev} t + \phi_e)$$

where Q is the tune ϕ_e is the starting phase of the betatron osc. y_{co} is the closed orbit position



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Betatron Motion.

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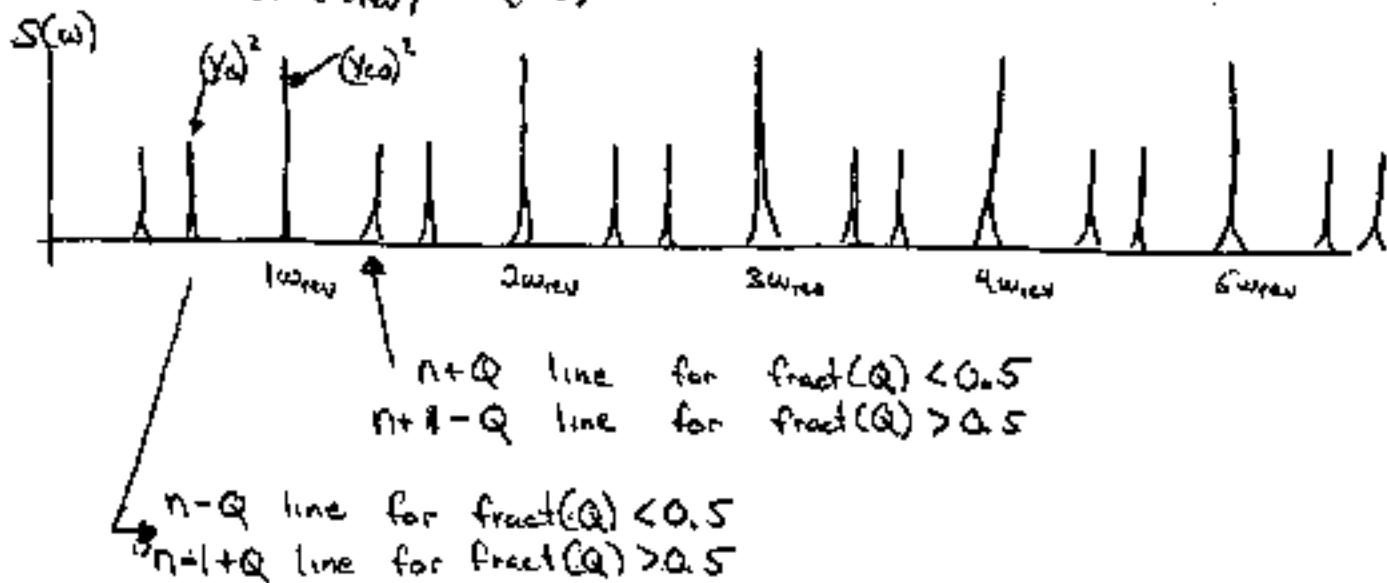
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$$V_{\text{out}} = \sqrt{2} \frac{q}{T_{\text{rev}}} Z_0 \sum_{n=-\infty}^{\infty} \left(\frac{Y_n}{d}\right) e^{jn\omega_{\text{rev}}t} + \frac{Y_a}{d} e^{j\omega_{\text{rev}}t} \cos(Q\omega_{\text{rev}}t + \phi_a)$$

$$e^{jn\omega_{\text{rev}}t} \cos(Q\omega_{\text{rev}}t + \phi_a)$$

$$= \frac{1}{2} e^{j\phi_a} e^{j(n+Q)\omega_{\text{rev}}t} + \frac{1}{2} e^{-j\phi_a} e^{j(n-Q)\omega_{\text{rev}}t}$$

$$S(\omega) = 2 \left(\frac{q}{T_{\text{rev}}}\right)^2 Z_0 \left(\frac{Y_a}{d}\right)^2 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_{\text{rev}}) \\ + \frac{1}{2} \left(\frac{q}{T_{\text{rev}}}\right)^2 Z_0 \left(\frac{Y_a}{d}\right)^2 \sum_{n=-\infty}^{\infty} \delta(\omega - (n+Q)\omega_{\text{rev}}) \\ + \frac{1}{2} \left(\frac{q}{T_{\text{rev}}}\right)^2 Z_0 \left(\frac{Y_a}{d}\right)^2 \sum_{n=-\infty}^{\infty} \delta(\omega - (n-Q)\omega_{\text{rev}})$$



From one pickup : you cannot distinguish the integer part of the tune.

you can't tell if the ^{fractional part} tune >.5 or <.5



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AM modulation.

AM modulation Signal.

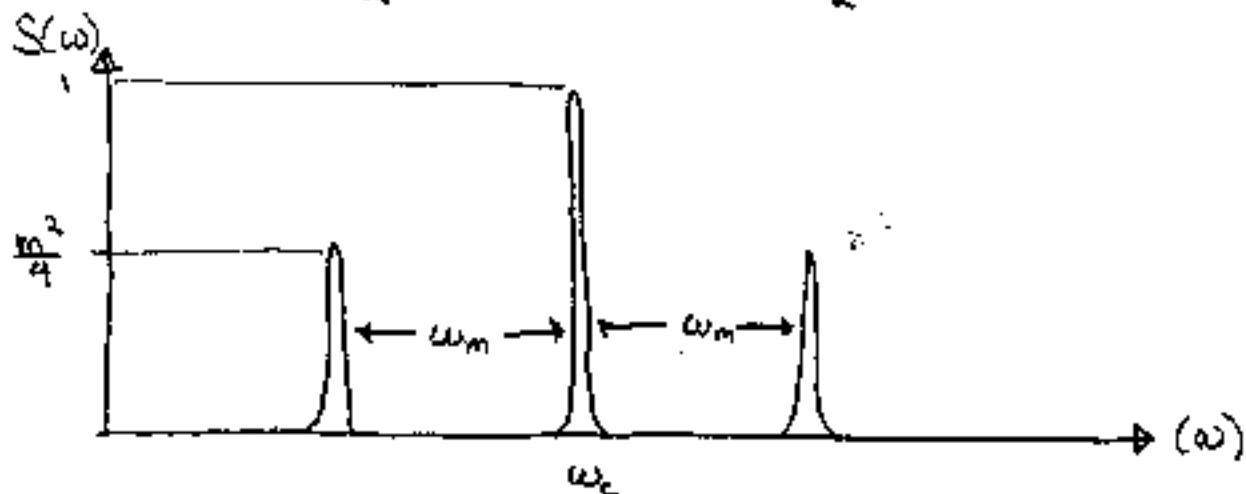
$$v(t) = (1 + m \cos \omega_m t) \cos \omega_c t$$

 ω_m is the modulation frequency ω_c is the carrier frequency

Using trig identity:

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$v(t) = \cos \omega_c t + \frac{m}{2} \cos((\omega_c + \omega_m)t) + \frac{m}{2} \cos((\omega_c - \omega_m)t)$$



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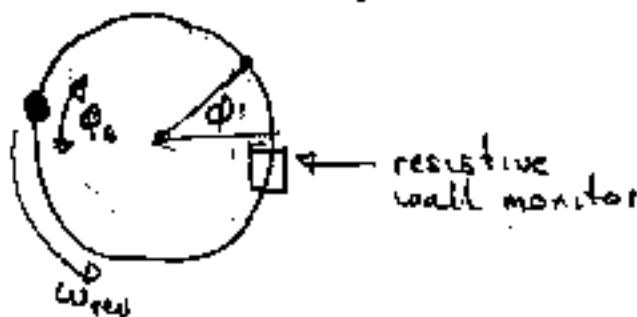
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Longitudinal Motion.

A single particle undergoes synchrotron oscillations ϕ_s



The particle's longitudinal position can be described by an azimuthal phase around the ring. The particle goes 2π radians in one trip around the ring

$$I(\phi) = \frac{q}{w_{rev}} \sum_{n=-\infty}^{\infty} \delta(\phi - 2n\pi)$$

This is a periodic function so we'll expand in a Fourier Series.

$$I(\phi) = \frac{q}{w_{rev}} \sum_{n=-\infty}^{\infty} e^{jn\phi}$$

$$\phi = \omega_{rev}t + \phi_s \sin(\Omega_s t + \alpha_s)$$

$$I(t) = \frac{q}{w_{rev}} \sum_{n=-\infty}^{\infty} e^{jn\omega_{rev}t} e^{jn\phi_s \sin(\Omega_s t + \alpha_s)}$$

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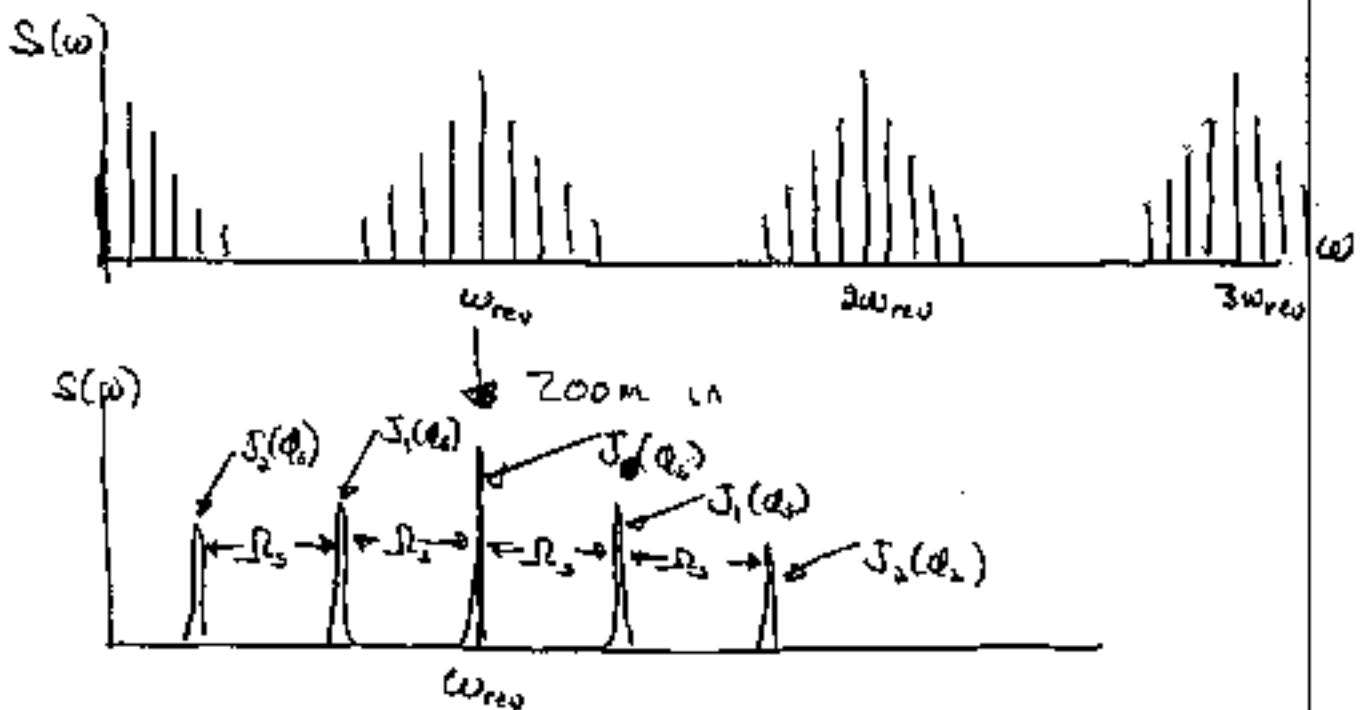
Longitudinal Motion

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$$e^{jz \sin(x)} = \sum_{m=-\infty}^{\infty} J_m(z) e^{jm x}$$

$$I(t) = \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_m(n\phi_s) e^{jm \omega_s t} e^{j(n\omega_{rev} + m\Omega_s)t}$$

Frequency Modulation

$$\omega = \omega_c + \omega_m \cos(\omega_s t)$$

$$\frac{d\omega}{dt} = \dot{\omega}$$

$$\phi = \omega_c t + \frac{\omega_m}{\omega_s} \sin(\omega_s t)$$

$$v(t) = \cos(\omega_c t + \frac{\omega_m}{\omega_s} \sin(\omega_s t))$$

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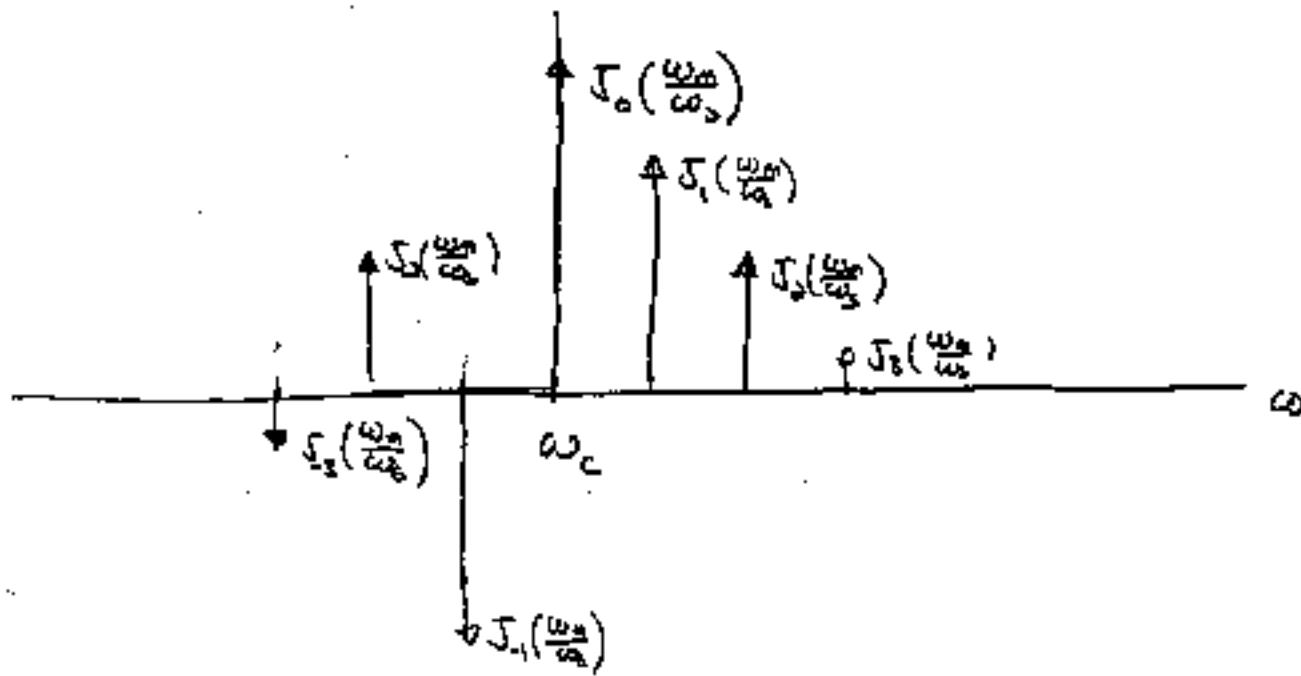
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Frequency Modulation.

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$$v(t) = \sum_{n=-\infty}^{\infty} J_n \left(\frac{\omega_m}{\omega_s} \right) \cos((\omega_c + n\omega_s)t)$$





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Multipole Distributions.

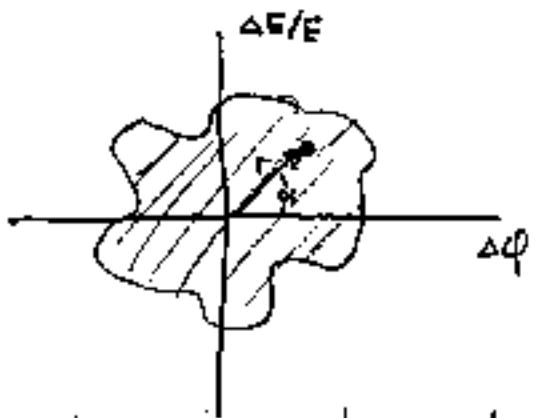
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The current for a single particle with synchrotron amplitude ϕ_s , and phase α_s is:

$$I_{sp}(\phi_s, \alpha_s) = \frac{q}{T_{rev}} \sum_n \sum_m J_m(n\phi_s) e^{j m \alpha_s} e^{j(n\omega_{rev} + m\Omega_s)}$$

What about a collection of particles in long. phase space



Each particle has a polar phase space coordinate r, α . The density in phase space must be periodic in α with a period of 2π .

$$\psi(r, \alpha) = f(r) \sum_{k=-\infty}^{\infty} c_k e^{jk\alpha}$$

$$N_p = \int_0^\infty r dr \int_0^{2\pi} d\alpha |\psi(r, \alpha)|^2$$

Also since $\psi(r, \alpha)$ must be real

$$c_k = c_{-k}^*$$



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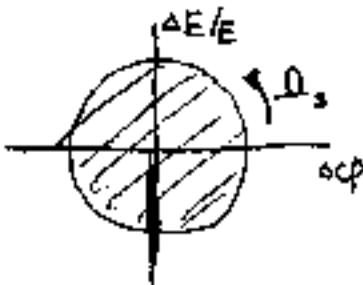
Multipole Distributions

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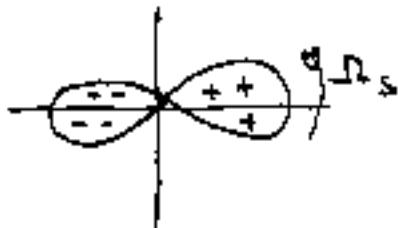
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The different values of k are multipoles.

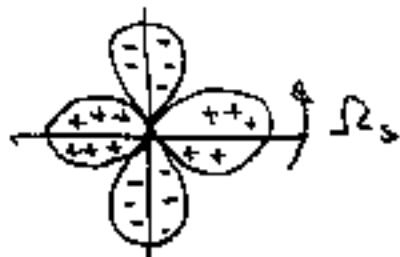
$k = 0$ monopole



$k = 1$ dipole

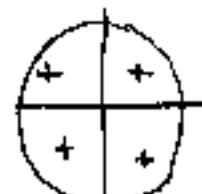
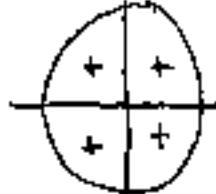
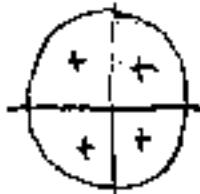
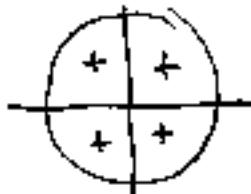
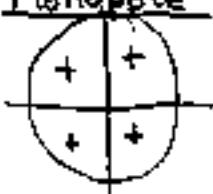


$k = 2$ quadrupole

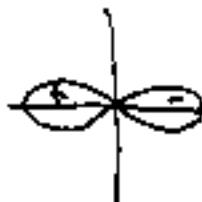
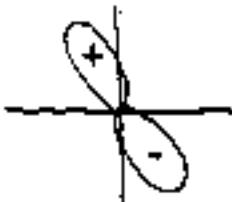
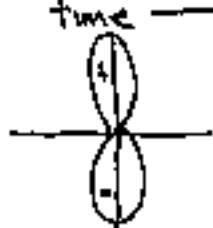
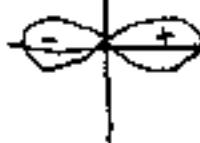


Each distribution spins with a frequency Ω_s

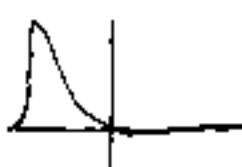
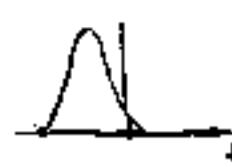
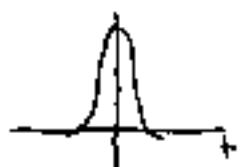
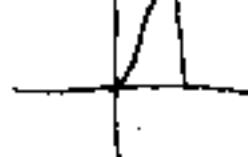
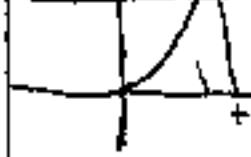
Monopole



Dipole



Time trace



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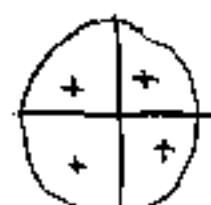
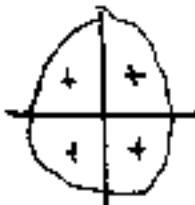
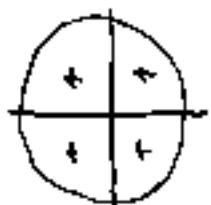
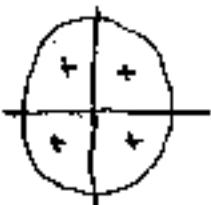
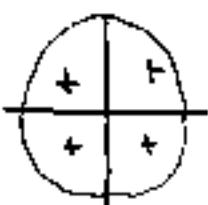
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Multipole Distributions

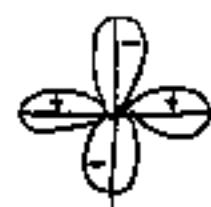
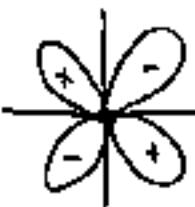
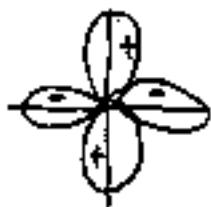
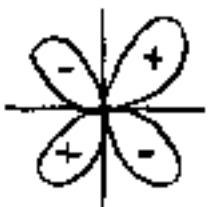
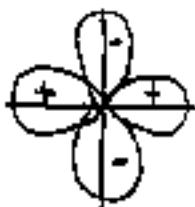
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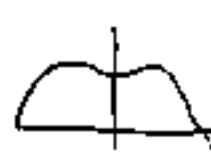
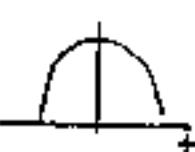
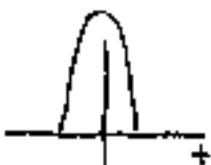
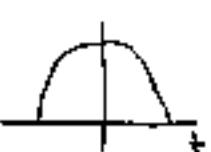
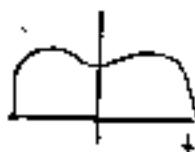
Monopole



Quadrupole



Time trace



The total current is:

$$I_T(t) = \int_0^{\infty} r dr \int_0^{2\pi} d\alpha \varphi(r, \alpha) I_p(r, \alpha, t)$$

$$I_T(t) = q \omega_{rev} \sum_n \sum_k F_k(n) C_k e^{j(n\omega_{rev} + k\beta_s)t}$$

where

$$F_k(n) = \int_0^{\infty} J_k(nr) f(r) r dr$$

is a frequency ($n\omega_{rev}$) form factor.



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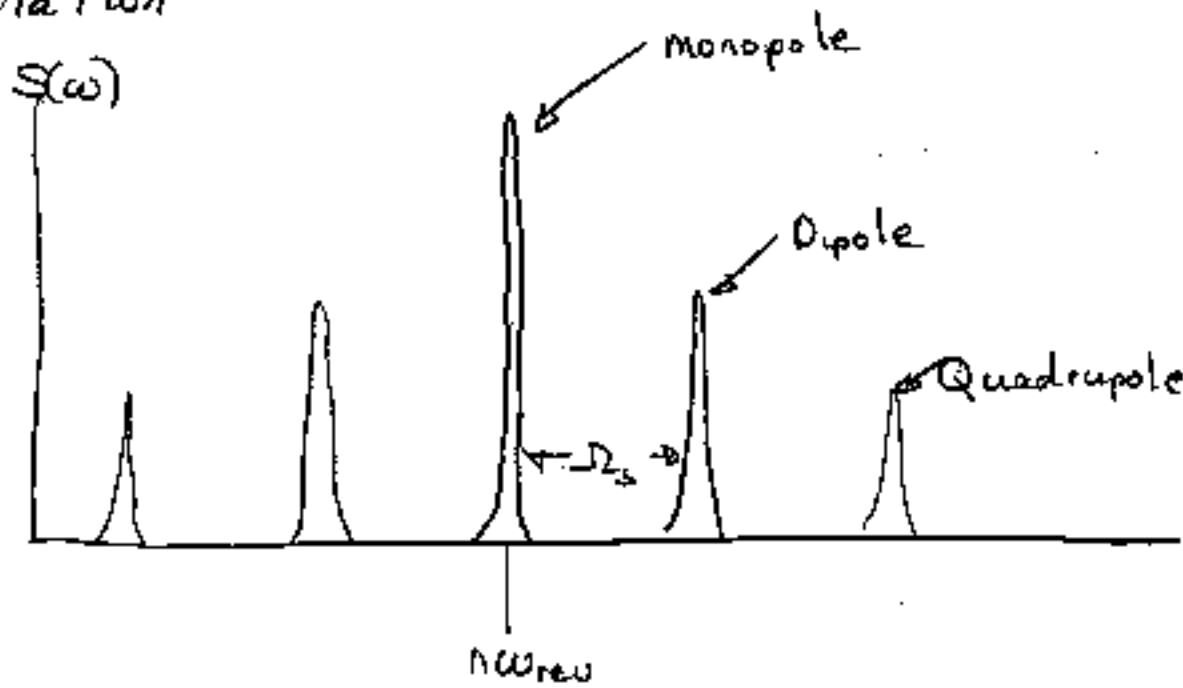
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Multipole Distributions

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Each synchrotron line in the spectrum corresponds to a different multipole mode oscillation



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RF Cavities

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For circular accelerators, the beam can only be accelerated by a time-varying (RF) electromagnetic field.

Faraday's Law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot d\vec{s}$$

The integral

$$q \oint_c \vec{E} \cdot d\vec{l}$$

is the energy gained by a particle (with charge) during one trip around the accelerator.

For a machine of fixed path (synchrotron),

if $\frac{\partial \vec{B}}{\partial t} = 0$ then $q \oint_c \vec{E} \cdot d\vec{l} = 0$



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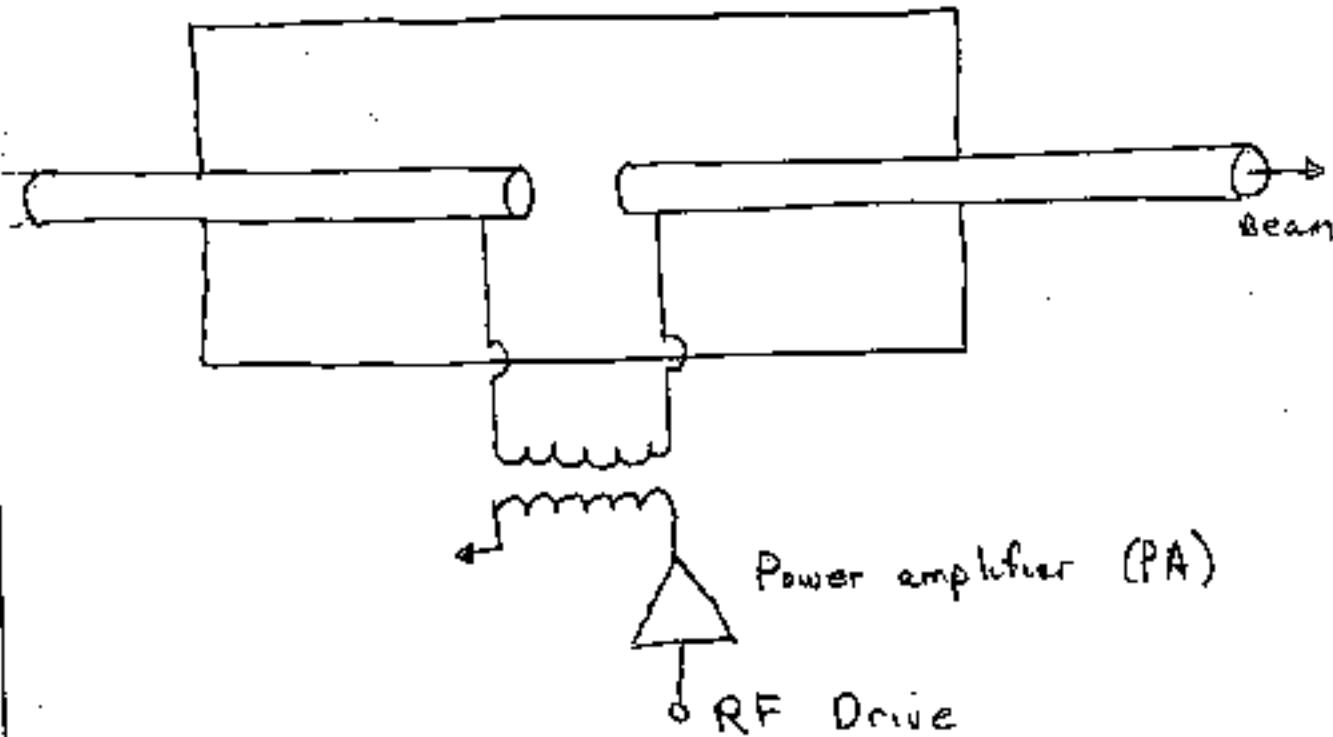
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RF Cavities

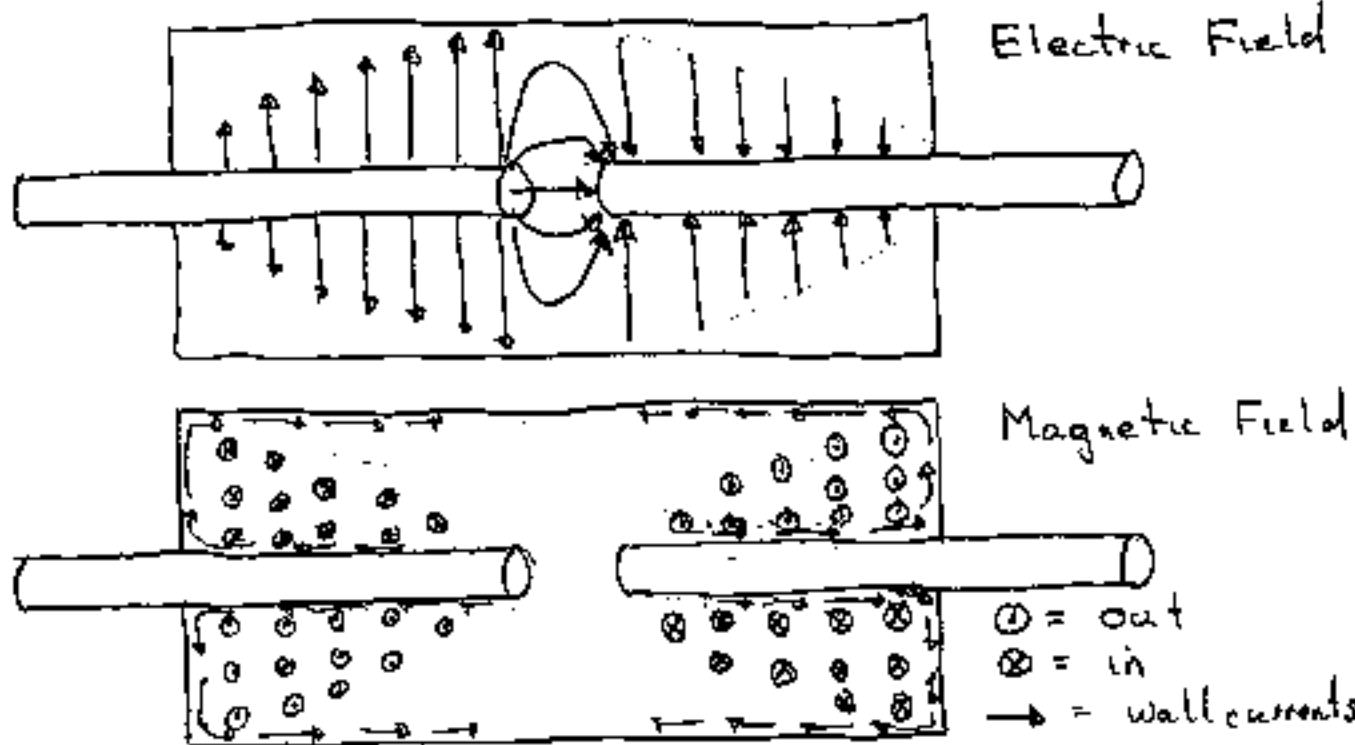
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Transmission line type cavity.



At one instant in time, for the fundamental mode.





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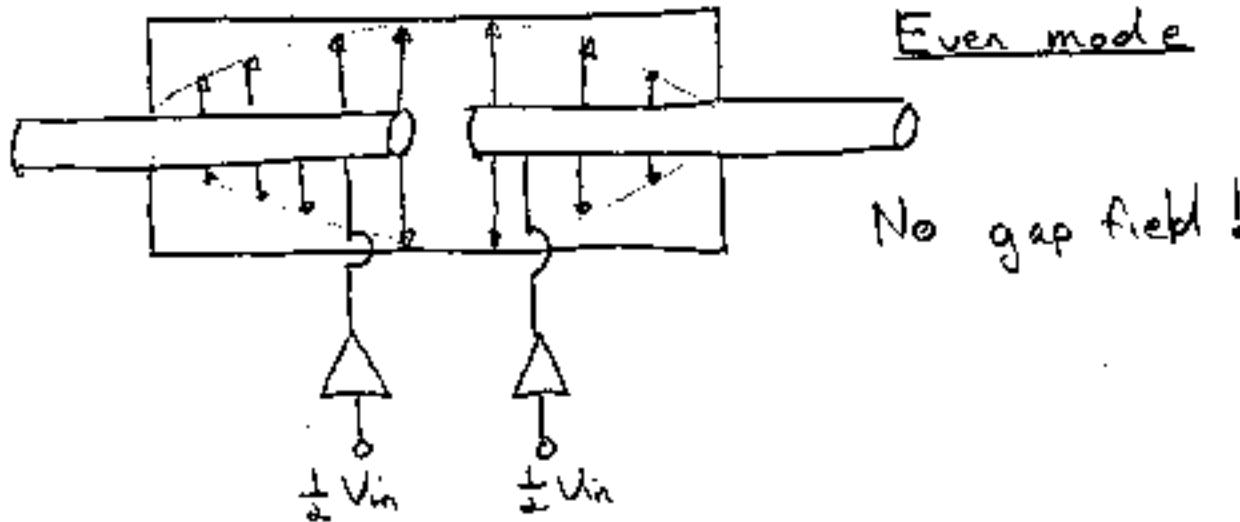
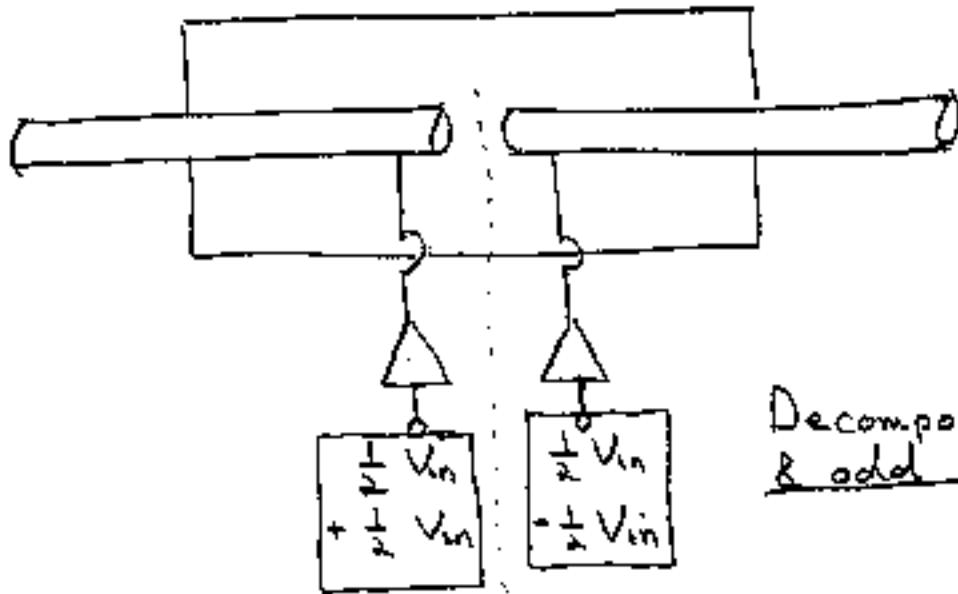
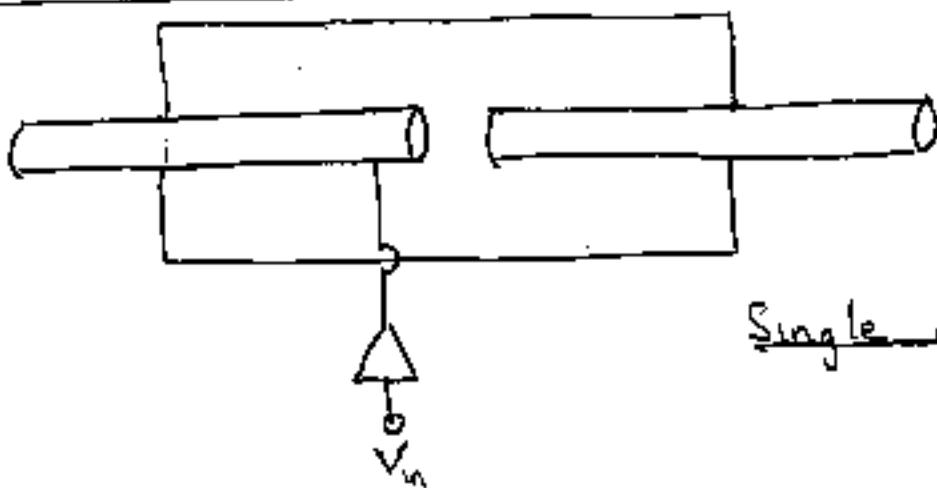
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RF Cavity Modes

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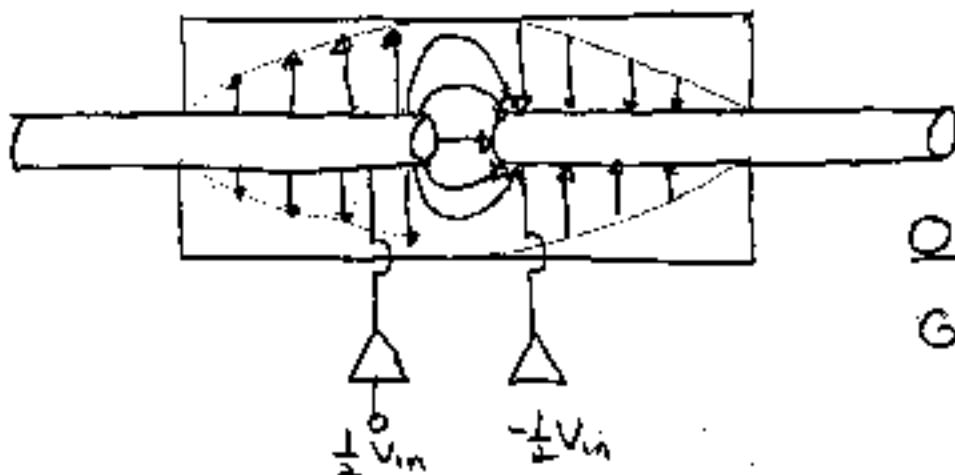
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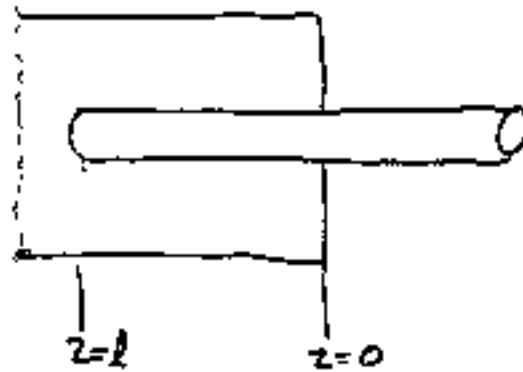
RF Cavity Modes

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Need to solve only $\frac{1}{2}$ of problem



Ignore gap capacitance

Cavity looks like shorted section of transmission line

$$V = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$

$$I = \frac{V^+}{Z_0} e^{-j\beta z} - \frac{V^-}{Z_0} e^{+j\beta z}$$

where Z_0 is the characteristic Impedance of the transmission line.

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RF Cavity modes.

Boundary Condition ① : At $z=0$ $V=0$

$$V = V_0 \sin \beta z$$

$$I = -j \frac{V_0}{Z_0} \cos \beta z$$

Boundary condition ② at $z=l$ $I=0$ only occurs when $\cos Bl = 0$

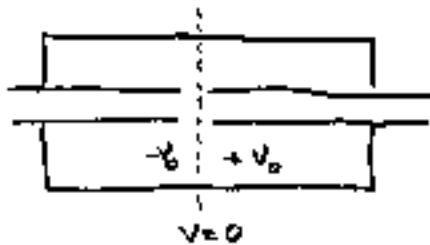
$$\beta l = (2n+1) \frac{\pi}{2} \quad n=0, 1, 2, \dots$$

$$f_n = (2n+1) \frac{c}{4l} \quad \text{or } l = (2n+1) \frac{\lambda}{4}$$

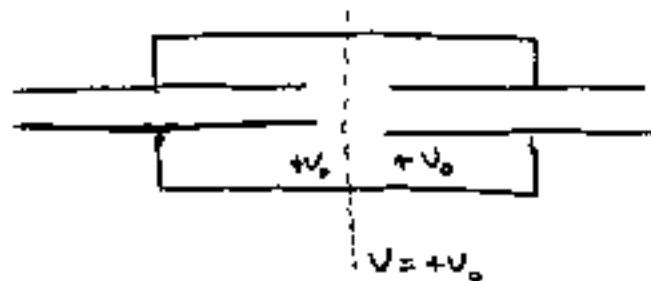
Different values of n are modes.

Even & odd decomposition have the same mode frequencies. Modes that occur at the same frequency are degenerate.

The even & odd modes can be split if we include the gap capacitance.



Odd mode



Even mode

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RF Cavity Modes

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In the even mode, since the voltage is the same on both sides of the gap, no capacitive current can flow across the gap.

In the odd mode;

Boundary condition ② at $z=l$

$$I = j\omega C_g V \quad \text{where } C_g \text{ is gap capacitance.}$$

$$\omega C_{gap} = \frac{1}{Z_0} \frac{\cos Bl}{\sin Bl}$$

Consider the first mode only ($n=0$) and a very small gap capacitance.

$$Bl = \frac{\pi}{2} + \Delta \quad (\Delta \ll 1)$$

$$\frac{\cos Bl}{\sin Bl} \approx -\Delta$$

$$\Delta \approx -\frac{\pi}{2} \frac{c}{l} C_g Z_0$$

$$\Delta f \approx -\frac{1}{4} \left(\frac{c}{l}\right)^2 C_g Z_0$$

The odd mode is shifted down in frequency.
The even mode remains unchanged.

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Cavity Q

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We have ignored any loss in the cavity. Using the transmission line type cavity as an example, assume a small resistive loss of $R_L \Omega/m$ in the walls of the cavity. The power loss is:

$$P_L = 2 \cdot \frac{1}{2} \int_{-l/2}^{l/2} |I(z)|^2 r_e dz$$

the surface

$$= \frac{1}{2} r_e l \left(\frac{V_0}{Z_0} \right)^2$$

The Quality factor (Q) of the cavity is defined as

$$Q = 2\pi \frac{\text{Energy stored in cavity}}{\text{Energy loss in one period}}$$
$$= \omega_0 \frac{\text{Electric Energy + Magnetic Energy}}{P_L}$$

Electric Energy

$$W_e = \frac{1}{4} \iiint_v \epsilon |E|^2 dv = 2 \cdot \frac{1}{2} \int_0^l C_s |V(z)|^2 dz$$

2 sides of cavity



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Cavity Q

where C_e is the capacitance / length

$$C_e = \frac{1}{c Z_0}$$

$$W_e = \frac{1}{4} \frac{1}{c Z_0} V_0^2$$

Magnetic energy

$$W_h = \frac{1}{4} \iiint \epsilon_0 |H|^2 dv = 2 \cdot \frac{1}{4} \int_0^L L_e |I(z)|^2 dz$$

where L_e is the inductance / length

$$L_e = \frac{Z_0}{c}$$

$$W_h = \frac{1}{4} \frac{1}{c Z_0} V_0^2$$

$$Q = \frac{\omega_0 \frac{1}{2} \frac{l}{c Z_0} V_0^2}{\frac{1}{2} \frac{l}{Z_0} \frac{V_0^2}{Z_0}} = \frac{\omega_0 l}{c} \frac{Z_0}{l}$$



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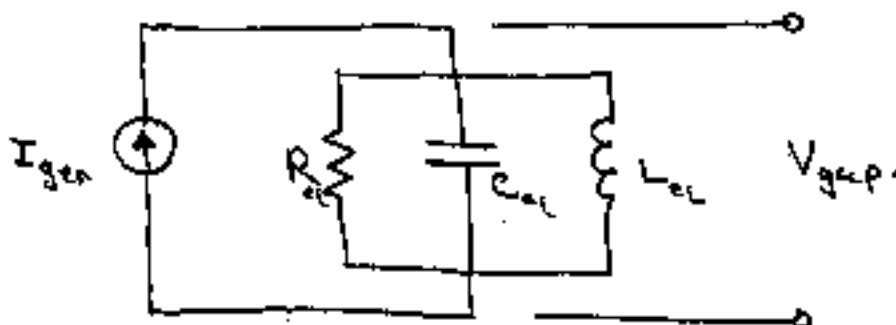
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RLC model for a Cavity mode.

For each mode, we can describe the cavity as a simple RLC circuit.



The R is a function of the energy lost.

The L is a function of the magnetic energy stored

The C is a function of the electric energy stored.

$$P_L = \frac{1}{2} \frac{V_{gap}^2}{R_{eq}}$$

For the transmission line cavity

$$R_{eq} = \frac{4Z_0^2}{r_{cav}} \quad (\text{remember } V_{gap} = 2V_0)$$

$$W_E = \frac{1}{4} C_{eq} V_{gap}^2$$

$$C_{eq} = \frac{1}{4} \frac{l}{c Z_0} \quad \text{for trans. line cavity}$$

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RLC model for a cavity mode

$$W_n = \frac{1}{4} L_{eq} (I_{max})^2 = \frac{1}{4} \frac{(V_{gap})^2}{\omega_0^2 C_{eq}}$$

$$L_{eq} = \frac{4 \frac{Z_0 l}{c}}{\left(\frac{\omega_0 l}{c}\right)^2} \quad \text{for transmission line model.}$$

The transfer function of the cavity is defined as

$$Z = \frac{V_{gap}}{I_{gen}}$$

$$Z = R_{eq} \frac{\frac{s}{R_{eq} C_{eq}}}{s^2 + \frac{s}{R_{eq} C_{eq}} + \frac{1}{L_{eq} C_{eq}}}$$

where $s = j\omega$

$$R_{eq} C_{eq} = \frac{l}{c} \frac{Z_0}{R_0 l} = \frac{Q}{\omega_0}$$

$$Q = \omega_0 R_{eq} C_{eq}$$

$$L_{eq} C_{eq} = \frac{1}{\omega_0^2}$$

$$\frac{R_{eq}}{Q} = \sqrt{\frac{L_{eq}}{C_{eq}}}$$

$$Z = R_{eq} \frac{s \frac{\omega_0}{Q}}{s^2 + \frac{s \omega_0}{Q} + \omega_0^2}$$



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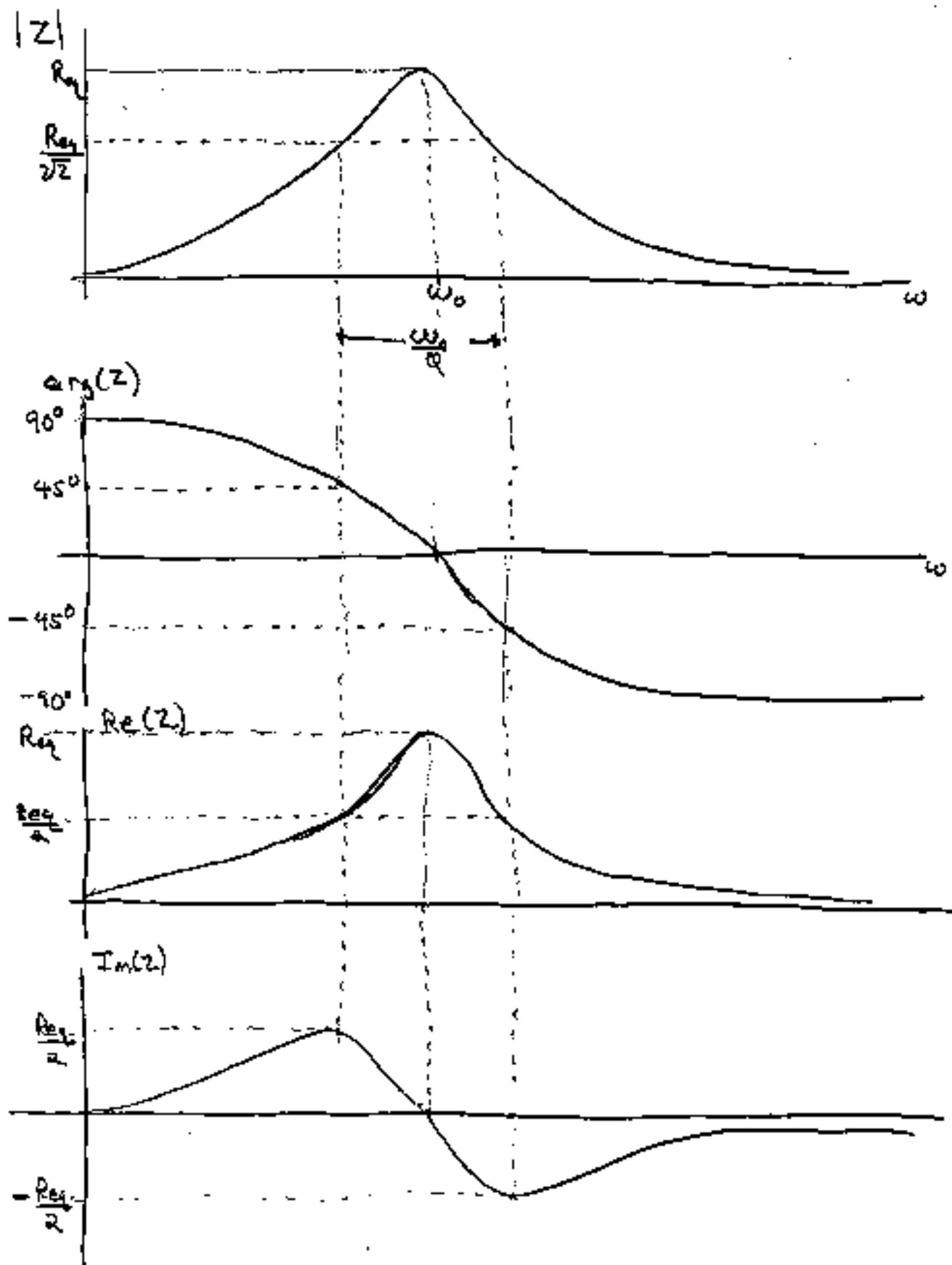
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RLC model for a cavity.

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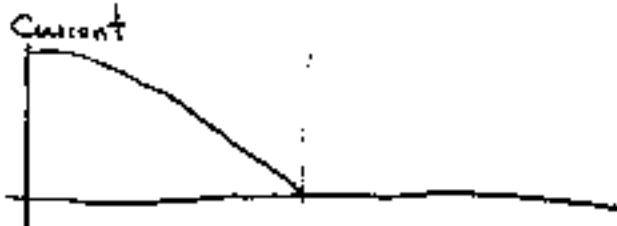
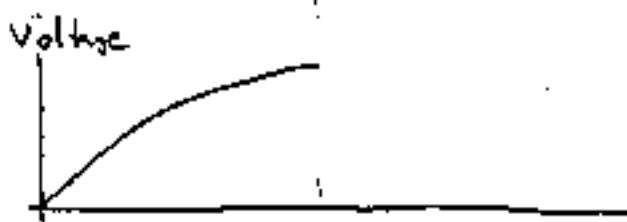
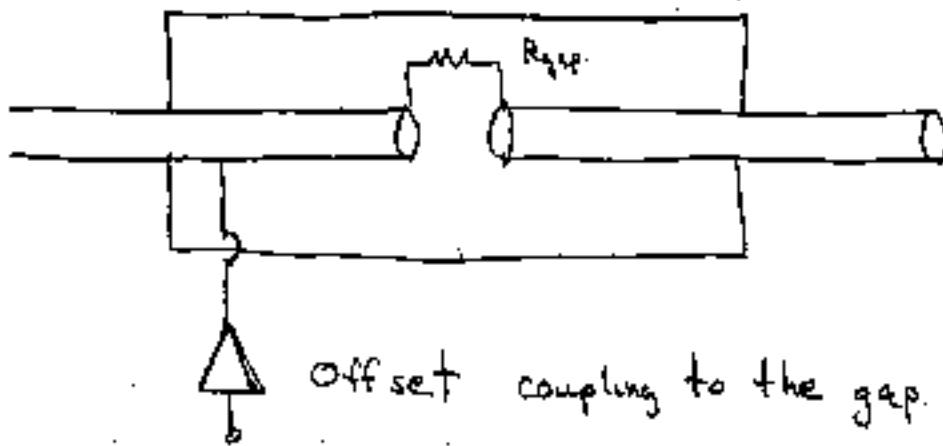
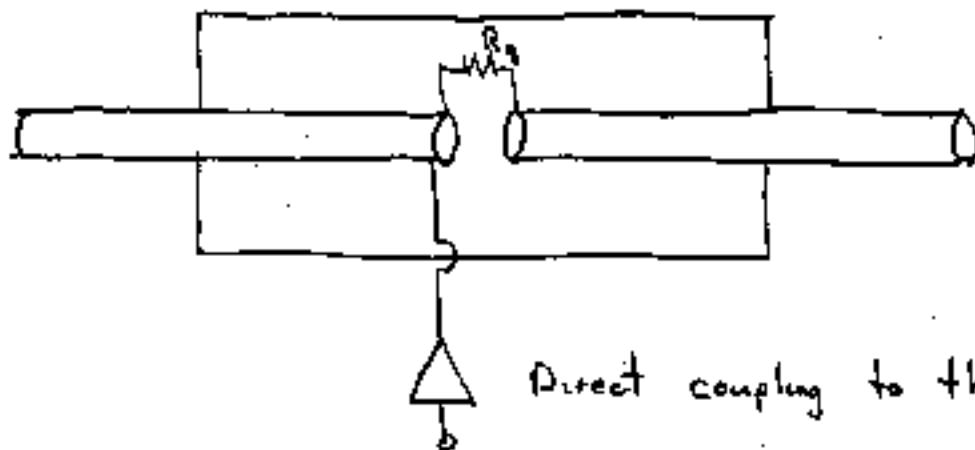
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Inductive Cavity Coupling.

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As the drive point is moved closer to the end of the cavity (away from the gap), the amount of current needed to develop a given voltage at the

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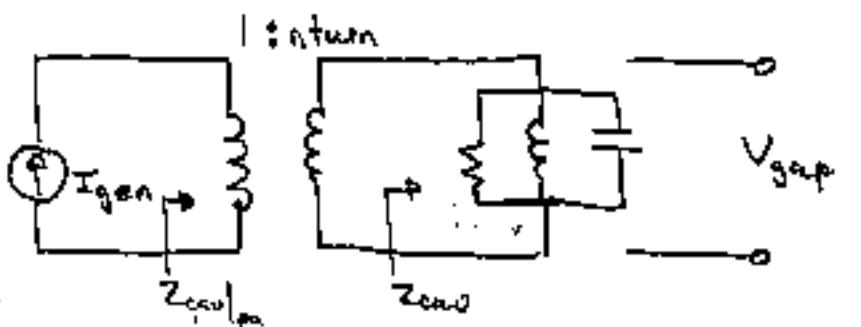
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Inductive Cavity Coupling.

gap must increase. Therefore the impedance of the cavity gap (at resonance) as seen by the power amp decreases as the drive point is moved away from the gap. We can model the drive point as a transformer



return increases as the drive point is moved away from the gap.

$$Z_{\text{cav}} \left|_{\text{PA}} \right. = \left(\frac{1}{\text{return}} \right)^2 Z_{\text{cav}}$$



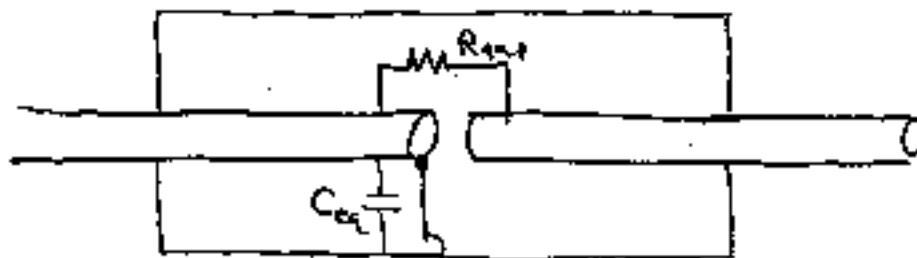
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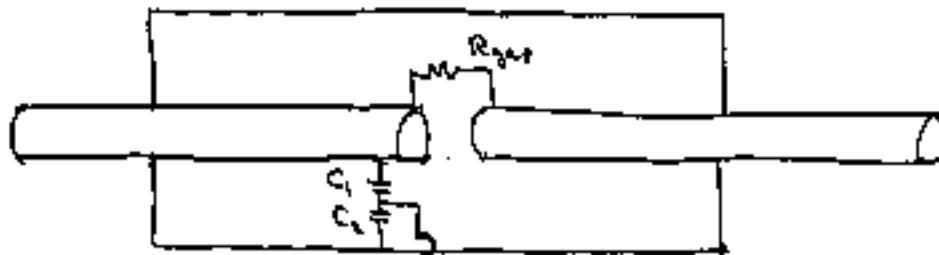
Capacitive Cavity Coupling.

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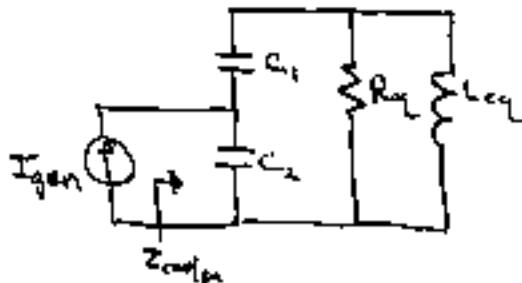


PA Direct Coupling to gap.



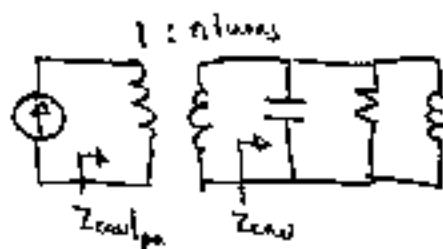
PA Capacitive coupling to gap.

Equivalent circuit.



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$Z_{pa} = \frac{1}{C_2/C} Z_{cav}$$



$$n \text{ turns} = C_2 / C_{eq}$$

As probe is pulled away from gap,
Impedance of the cavity as seen
from the PA decreases.



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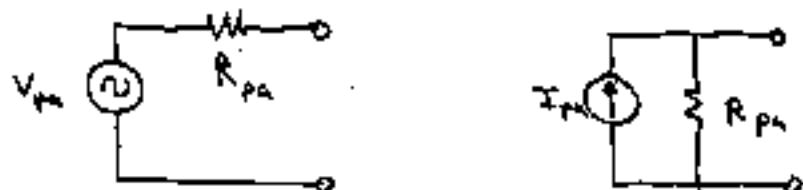
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Loaded Q of Cavities.

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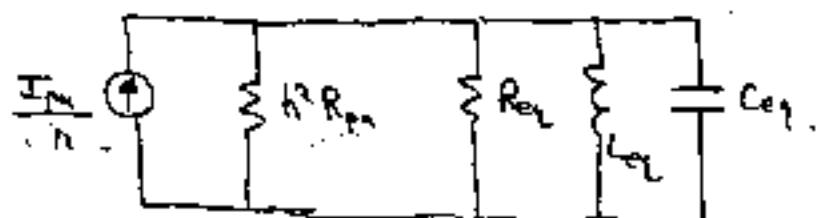
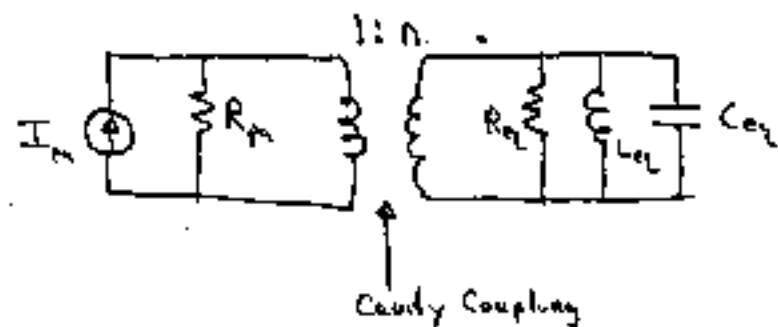
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Power Amplifier.



$$V_{pa} = I_{pa} R_{pa} \quad (\text{Thevenin's equivalent})$$

Total Cavity Circuit.



$$\frac{1}{R_T} = \frac{1}{R_{eq}} + \frac{1}{n^2 R_{pa}}$$

$$\text{unloaded } Q_0 = \omega_0 R_{eq} C_{eq}$$

$$\text{loaded } Q_L = \omega_0 R_T C_{eq}$$

$$\text{external } Q_{ext} = \omega_0 n^2 R_{pa} C_{eq}$$



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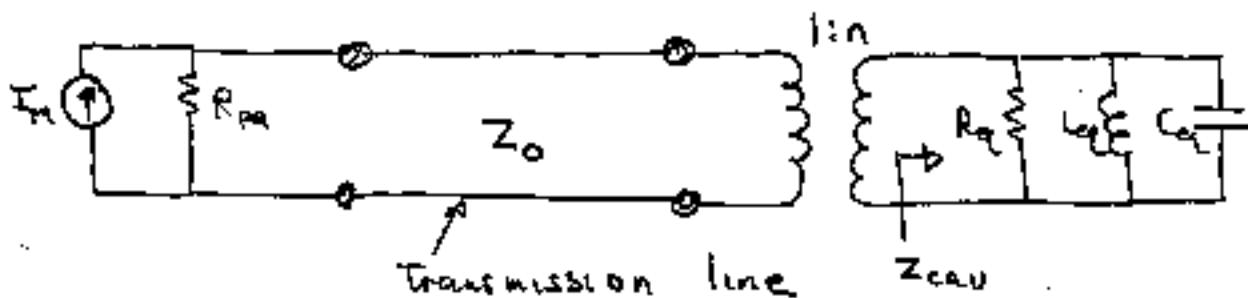
Loaded Q of Cavities

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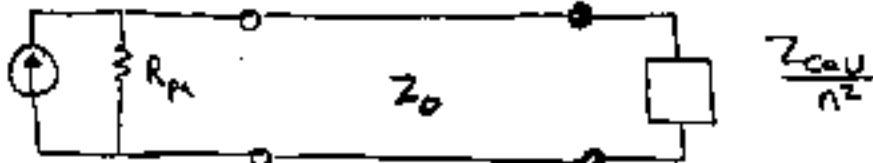
$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}$$

Attaching power amplifier loads the circuit down and de-Q's it.

Cavity Coupling

Power source is usually matched to transmission line
i.e. $R_{pa} = Z_0$

Bringing Cavity impedance over to generator side.



$$\text{Coupling Parameter } r = \frac{R_q / n^2}{Z_0}$$

- $r < 1$ under coupled
- $r = 1$ critically coupled
- $r > 1$ over coupled.

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Measuring Cavity Coupling.

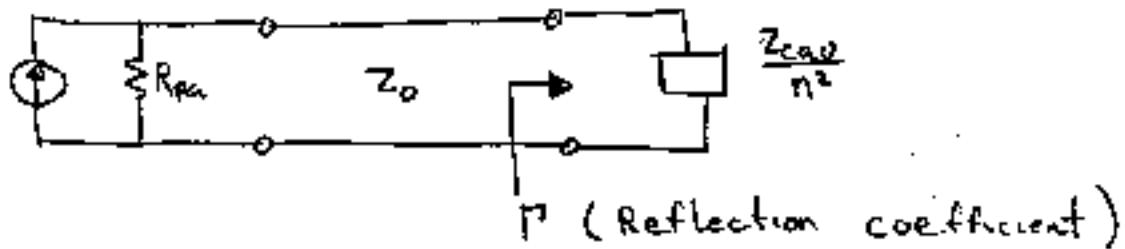
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$$Z_{\text{cav}}(\omega) = \frac{j \omega_0 R_{\text{eq}}}{Q} \frac{\omega_0^2 - \omega^2 + j \omega \omega_0}{\omega_0^2 - \omega^2 + j \omega \omega_0}$$

$$Z_{\text{cav}}(\phi) = R_{\text{eq}} \cos \phi e^{-j\phi}$$

$$\phi = \tan^{-1} \left(\frac{\omega_0^2 - \omega^2}{\frac{\omega \omega_0}{Q}} \right)$$



$$\Gamma = \frac{\frac{Z_{\text{cav}}}{n^2} - Z_0}{\frac{Z_{\text{cav}}}{n^2} + Z_0}$$

$$\Gamma = \frac{r \cos \phi e^{j\phi} - 1}{r \cos \phi e^{j\phi} + 1}$$

$$\Gamma = \frac{(r^2 \cos^2 \phi - 1) + j(2r \cos \phi \sin \phi)}{r(r+2) \cos^2 \phi + 1}$$



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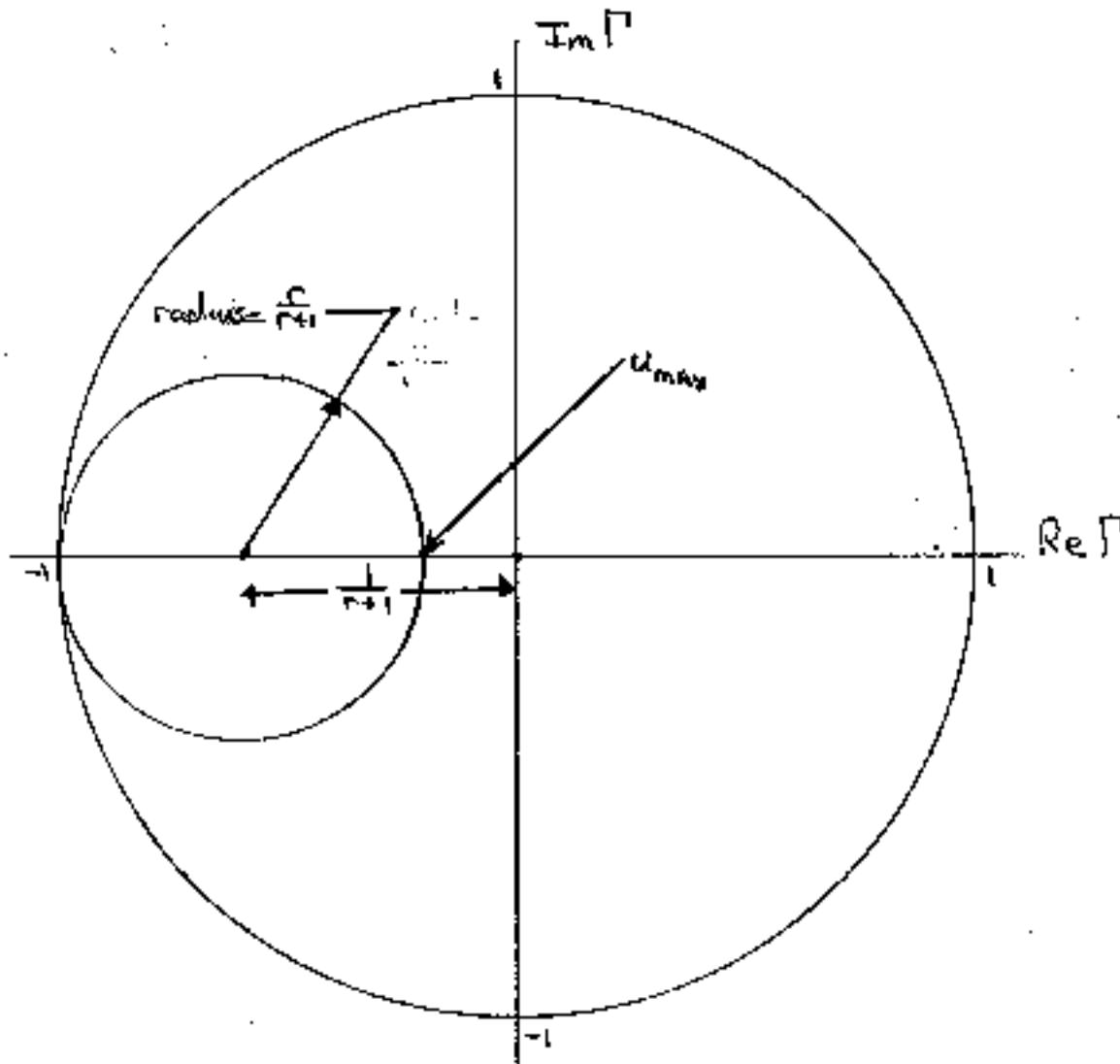
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This equation defines a circle on the Γ plane



Radius of circle at $-\frac{r}{r+1}$

Center located at $\Gamma = \left(-\frac{1}{r+1}, 0\right)$

Left edge of circle located at $\Gamma = (-1, 0)$

Right edge of circle located at $\Gamma = \left(\frac{r-1}{r+1}, 0\right)$

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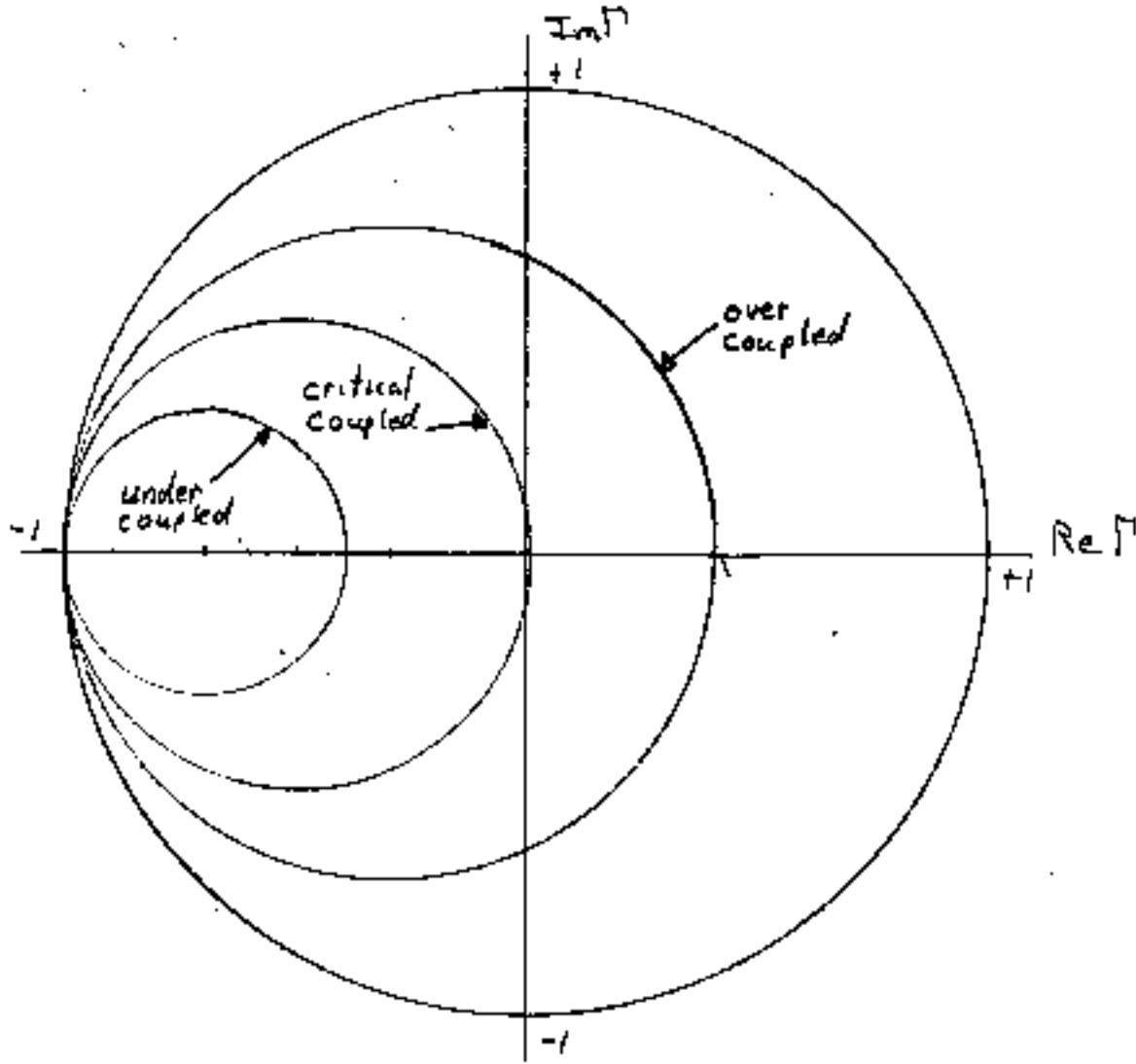
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Measure the reflection coefficient at the right hand side of the circle

$$\Gamma_{RHE} = u_{max} + j\theta$$

$$r = \frac{1 + u_{max}}{1 - u_{max}} = \text{coupling}$$

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Measuring the unloaded Q of Cavity

$$\text{At } \omega = \omega_0 - \frac{\omega_0}{Q_0}$$

$$\phi = \frac{\pi}{4}$$

$$\text{Im}(Z_{\text{cav}}) = \text{Re}(Z_{\text{cav}})$$

$$\text{At } \omega = \omega_0 + \frac{\omega_0}{Q_0}$$

$$\phi = -\frac{\pi}{4}$$

$$\text{Im}(Z_{\text{cav}}) = -\text{Re}(Z_{\text{cav}})$$

$$\text{For } \phi = \frac{\pi}{4} \quad \Gamma = \frac{r^2 - 2}{r(r+2) + 2} + i \frac{2r}{r(r+2) + 2}$$

If we let "r" be a running parameter, this equation traces out a circle:

$$u^2 + (v+1)^2 = 2 \quad \text{where } \Gamma = u + iv$$

$$\text{for } \phi = -\frac{\pi}{4} \quad \Gamma = \frac{r^2 - 2}{r(r+2) + 2} - i \frac{2r}{r(r+2) + 2}$$

Which traces out a circle:

$$u^2 + (v-1)^2 = 2 \quad \text{where } \Gamma = u + iv$$

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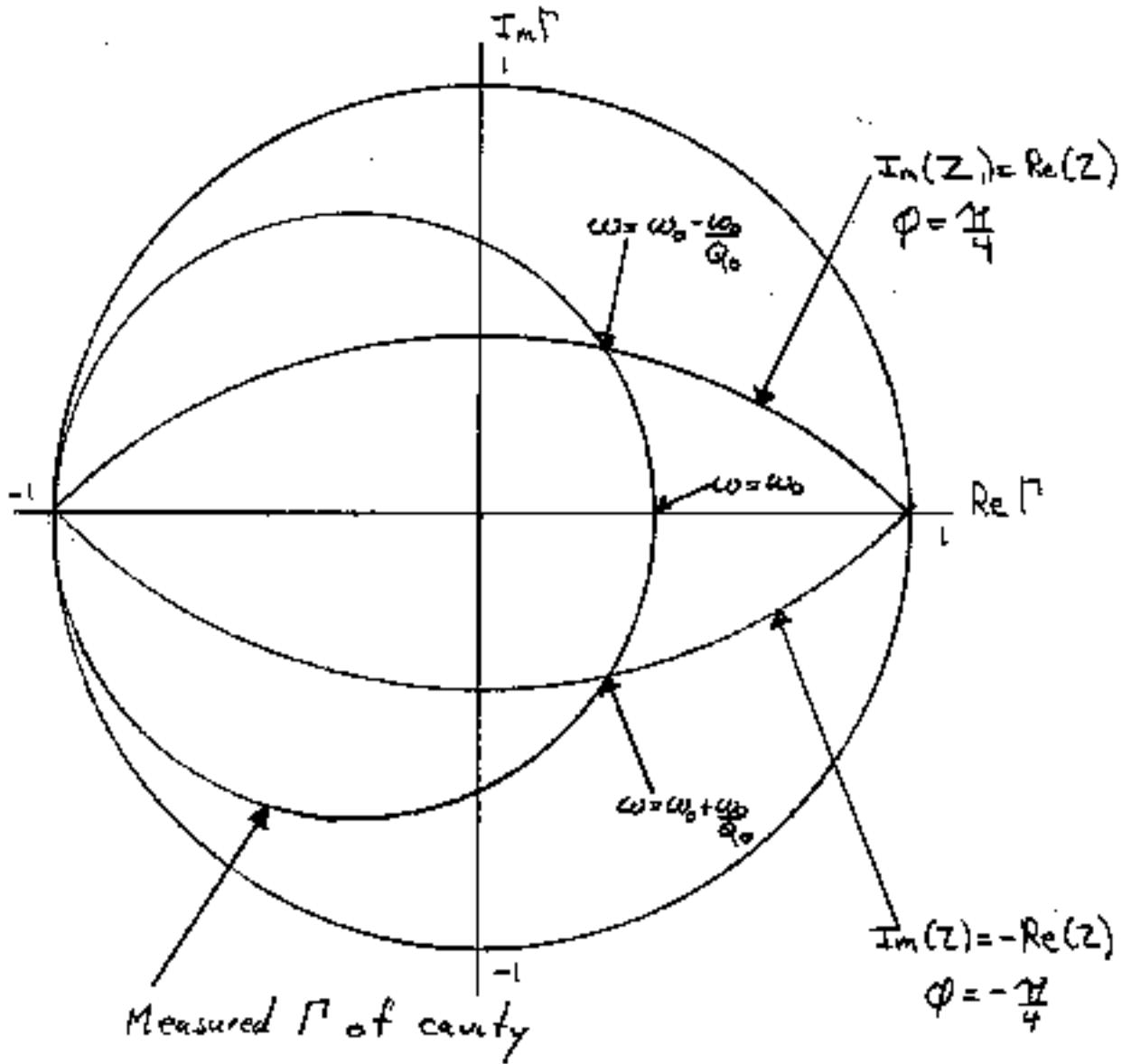
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Measuring the unloaded Γ of cavities

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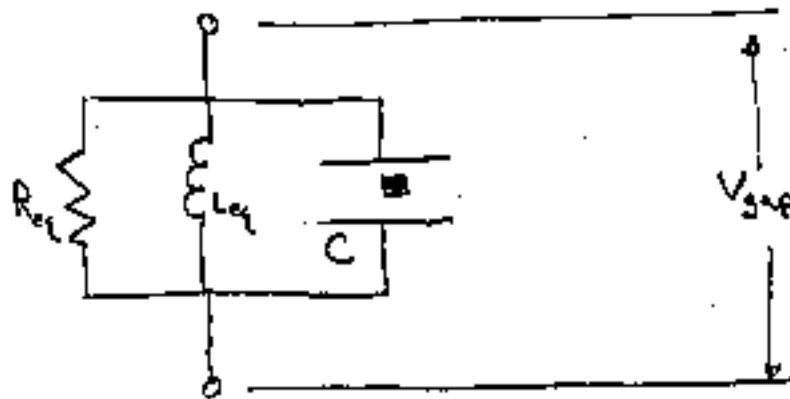
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In the capacitor, place a small dielectric cube. The stored energy in the capacitor will change. Assume that the small cube will not distort the field appreciably.

$$W_E = \frac{1}{4} C_{eq} V_{gap}^2 - \frac{1}{4} \epsilon_0 E_c^2 dv + \frac{1}{4} \epsilon_r \epsilon_0 E_c^2 dv$$

↑
Total original ↑
Electric energy original Electric
 energy in cube new Electric
 energy in cube

where E_c is the electric field in the cube

dv is the volume of the cube

$$W_E = \frac{1}{4} \iiint \epsilon_0 E^2 dv$$

$$W_E = \frac{1}{4} CV^2 = \frac{1}{4} (C_{eq} + \Delta C) V^2$$

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$$\Delta C = \epsilon_0 (\epsilon_r - 1) d\pi \left(\frac{E_p}{V} \right)^2$$

The resonant frequency of the cavity will shift

$$(\omega_0 + \Delta\omega)^2 = \frac{1}{L_C (C_{eq} + \Delta C)}$$

for $\Delta\omega \ll \omega_0$ & $\Delta C \ll C_{eq}$

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{2} \frac{\Delta C}{C_{eq}}$$

$$= \frac{1}{2} \frac{\epsilon_0 (\epsilon_r - 1) d\pi \left(\frac{E_p}{V} \right)^2}{C_{eq}}$$

$$= \frac{1}{2} \frac{\epsilon_0 (\epsilon_r - 1) d\pi E_p^2}{C_e V_g^2}$$

$$\boxed{\frac{\Delta\omega}{\omega_0} = \frac{\Delta W_E}{W_T}}$$

Had we used a magnetic bead ($\mu_r > 1$) or a metal bead.

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta W_E - \Delta W_M}{W_E + W_M}$$

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Also, the shape of the bead will distort the field in the vicinity of the bead so a geometrical form factor must be used.

For a small dielectric bead $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_0$, radius = a

$$\frac{\Delta\omega}{\omega_0} = -\frac{\pi a^3}{W_T} \epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right) E_b^2$$

where E_b is field at the bead.

For a small metal bead with radius a .

$$\frac{\Delta\omega}{\omega_0} = -\frac{\pi a^3}{W_T} \left[\epsilon_0 E_b^2 + \frac{\mu_0 H_b^2}{2} \right]$$

where E_b & H_b are the fields at the location of the bead before the bead is put into the cavity.

Note: A metallic bead can be used to measure the E field only if the bead is placed in a region where the magnetic field is zero.

In general, the shift in frequency is proportional to a form factor Φ



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$$\frac{\Delta\omega_a}{\omega_0} = -\frac{f_r E^2}{W_r}$$

$f_r = \pi a^3 \epsilon_0 \left(\frac{\epsilon_r - 1}{2r + 2} \right)$ for dielectric spherical bead

$f_r = \pi a^3 \epsilon_0$ for a metal spherical bead.

From the definition of cavity Q

$$Q = \omega_0 \frac{W_r}{P_r}$$

$$P_r = \frac{1}{2} \frac{V_{gap}}{R_{eq}}^2$$

$$E(x, y, z) = \left(\frac{1}{f_r} \frac{1}{2\omega_0} \frac{1}{R_{eq}/Q} \right)^{1/2} \left(\frac{\Delta\omega_a(x, y, z)}{\omega_0} \right)^{1/2}$$

where

$\frac{\Delta\omega_a}{\omega_0}(x, y, z)$ is the frequency shift due to the bead located at x, y, z

Since $\int_{gap} E(x, y, z) dz = V_{gap}$

$$\frac{R_{eq}}{Q} = -\frac{1}{f_r} \frac{1}{2\omega_0} \left[\int_{gap} \left(\frac{\Delta\omega_a(x, y, z)}{\omega_0} \right)^{1/2} dz \right]^2$$

To include a beam transit time effect

$$\frac{R_{eqT}}{Q} = \frac{1}{f_r} \frac{1}{2\omega_0} \left[\int_{gap} \left(\frac{\Delta\omega_a(x, y, z)}{\omega_0} \right)^{1/2} \cos \left(\frac{\omega_0}{v_{beam}} z \right) dz \right]^2$$